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Part II
Electromagnetic wave propagation

Contents

1 Electromagnetic waves

1.1 Maxwell Equations

- In the MKS system (or S.I.), the equations of electrodynamics are, :

$$\begin{aligned}
\vec{\nabla} \cdot \vec{D} &= \rho, && (1)\\
\vec{\nabla} \cdot \vec{B} &= 0, && (2)\\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, && (3)\\
\vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}. && (4)
\end{aligned}$$

- For linear media, $\vec{D} = \epsilon \vec{E}$ y $\vec{B} = \mu \vec{H}$.
- In vacuum, $\epsilon = \epsilon_\circ$ and $\mu = \mu_\circ$.

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1.2 Electrodynamic potentials

- Since $\vec{\nabla} \cdot \vec{B} = 0$, we have
$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{5}$$
- For $\vec{E}$, we use Eq. 5 and Eq. 3:
$$\vec{\nabla} \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)}_{\equiv -\vec{\nabla}\Phi} = 0, \;\; \Rightarrow$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{6}$$

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1.3 Wave equations

- We want to write equations that determine the electrodynamic potentials $\vec{A}$ and Φ.
- Using the Maxwell Equations in vacuum to connect directly with $\vec{E}$ and $\vec{B}$, we have
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_\circ} \;\Rightarrow\; \nabla^2\Phi + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\rho}{\epsilon_\circ}, \tag{7}$$

$$\vec{\nabla} \times \frac{1}{\mu_\circ}\vec{B} = \vec{J} + \frac{1}{\epsilon_\circ}\frac{\partial \vec{E}}{\partial t} \;\Rightarrow$$
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\underbrace{\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)}_{\text{term for the Lorentz condition}} = -\mu_\circ \vec{J}. \tag{8}$$

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- If the term highlighted in Eq. 8 is null, which is called the *Lorentz Condition*,
$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0, \tag{9}$$
then we recover the wave equation for the potentials:
$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_\circ}, \tag{10}$$
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_\circ \vec{J}. \tag{11}$$

- To fulfill the Lorentz condition, we use the freedom of gauge:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \tag{12}$$

which leaves invariant $\vec{B} = \vec{\nabla} \times \vec{A}$.
- To also preserve $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t$, it is necessary that

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}. \tag{13}$$

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- If $\vec{A}$ and Φ both fulfill the general potential equations (Eqs. 8 and 7), but do not fulfill the Lorentz condition, then we can search for $\Lambda(\vec{x}, t)$ so that $\vec{A}'$ and Φ' do satisfy the Lorentz condition.
- Injecting Eqs. 12 and 13 in Eq. 9, we reach an equation for $\Lambda(\vec{x}, t)$:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t} = 0, \tag{14}$$

which is essentially a wave equation with a source term, i.e. exactly the type of equations that we will propose solutions for.

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- Independently of the Lorentz condition, we can manipulate the Maxwell equations to reach (tarea):

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\epsilon_\circ}\left(-\vec{\nabla}\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right), \tag{15}$$

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\mu_\circ\vec{\nabla} \times \vec{J}, \tag{16}$$

which are both wave equations with source terms.
- Away from the sources, i.e. in vacuum, both equations become the homogeneous wave equation.

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1.4 Poynting's theorem

- The power exerted by the electromagnetic force $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$ on a single charge q with velocity $\vec{v}$ is $\vec{v} \cdot \vec{\mathcal{F}} = q\vec{v} \cdot \vec{E}$.
- The power exerted on the charge density distribution ρ and on the current density distribution $\vec{J} = \rho\vec{v}$ inside a volume $d\mathcal{V}$ is thus

$$dP = \vec{J} \cdot \vec{E}\, d\mathcal{V}$$

- The total power exerted by the $(\vec{E}, \vec{B})$ field on the charges inside a volume $\mathcal{V}$ is

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x. \tag{17}$$

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- We want to connect P with the energy stored in the fields. Using the Ampère-Maxwell equation (Eq. 4) we solve for $\vec{J}$, and following standard handling (tarea),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \tag{18}$$

- Now with the induction law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Eq. 3),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \tag{19}$$

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- Remembering that for a linear medium $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H} \cdot \vec{B})$, and $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E} \cdot \vec{D})$, we reach

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x = -\int_{\mathcal{V}} \left[\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right] d^3x, \tag{20}$$

where we recognize

$$u = \frac{1}{2}\vec{E} \cdot \vec{D} + \frac{1}{2}\vec{B} \cdot \vec{H}, \tag{21}$$

and

$$\vec{S} = \vec{E} \times \vec{H}. \tag{22}$$

- For any volume $\mathcal{V}$, we conclude that

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}. \tag{23}$$

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- In the same way as for energy conservation, Eq. 23, we can also write the equation for the conservation of linear momentum. Newton's 2nd law for the variation of linear momentum $\delta \vec{p}_{\text{mec}}$ inside a volume $\delta\mathcal{V}$ is:

$$\frac{d\,\delta \vec{p}_{\text{mec}}}{dt} = \rho \vec{E} \delta\mathcal{V} + \rho \vec{v} \times \vec{B} \delta\mathcal{V}. \tag{24}$$

- In total,

$$\frac{d\vec{p}_{\text{mec}}}{dt} = \int_{\mathcal{V}} d^3x (\rho \vec{E} + \rho \vec{v} \times \vec{B}). \tag{25}$$

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- Using Maxwell's equation to replace ρ and $\vec{J}$, we reach (tarea):

$$\frac{d}{dt}(\vec{p}_{\rm mec} + \vec{p}_{\rm fields})|_i = \sum_j \int_{\mathcal{V}} d^3x \frac{\partial T_{ij}}{\partial x_j}, \tag{26}$$

with the following notations:

$$\vec{p}_{\rm fields} = \int \epsilon_\circ (\vec{E} \times \vec{B}) d^3x = \frac{1}{c^2} \int d^3x \vec{S}, \tag{27}$$

which we associate to the momentum in the fields since it fulfills a similar role as $\vec{p}_{\rm mec}$, and

$$T_{ij} = \epsilon_\circ \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \left(\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{ij} \right], \tag{28}$$

which is the tensor of electromagnetic tensions. .13

- For each component i the integrand of T_{ij} can be seen as a divergence, so

$$\frac{d}{dt}(\vec{p}_{\rm mec} + \vec{p}_{\rm fields})|_i = \oint_{\mathcal{S}} \sum_j T_{ij} n_j d\mathcal{A}, \tag{29}$$

where we recognize a flux integral over the surface bounding the volume $\mathcal{V}$. .14

2 Wave propagation in vacuum

2.1 Spectral decomposition

- In the absence of sources, if we decompose

$$\vec{E}(\vec{x}, t) = \frac{1}{2\pi} \int d\omega \vec{E}(\vec{x}, \omega) e^{iwt}, \tag{30}$$

the Maxwell equations yield

$$(\nabla^2 + \mu\epsilon\omega^2) \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} = 0. \tag{31}$$

- If ϵ and μ are both real, the solutions are $e^{\pm ikx}$, with $k = \sqrt{\mu\epsilon}\omega$
- We define the phase velocity $v_\phi = \frac{\omega}{k} = \frac{c}{n}$, where $n = \sqrt{\frac{\mu\epsilon}{\mu_\circ \epsilon_\circ}}$ is the refraction index.
- In general,

$$\left\{ \begin{array}{c} E_i \\ B_i \end{array} \right\} = \frac{1}{2\pi} \int d\omega \left\{ \begin{array}{c} \mathcal{E}_i \\ \mathcal{B}_i \end{array} \right\} e^{\pm i\vec{k}\cdot\vec{x} - iwt}, \tag{32}$$

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- We recognize d'Alembert's solution for the wave equation,

$$\left\{ \begin{array}{c} E_i \\ B_i \end{array} \right\} = \frac{1}{2\pi} \int d\omega \left\{ \begin{array}{c} \mathcal{E}_i \\ \mathcal{B}_i \end{array} \right\} e^{\pm ik(\hat{n}\cdot\vec{x}-v_\phi t)}, \tag{33}$$

where each component i has a form $f(\hat{n}\cdot\vec{x}-v_\phi t)+g(\hat{n}\cdot\vec{x}+v_\phi t)$, and where $\hat{n}$ is the direction of propagation.
- Using Maxwell's equations (tarea), $\hat{n}\cdot\vec{\mathcal{E}}=0$, $\hat{n}\cdot\vec{\mathcal{B}}=0$ and $\vec{\mathcal{B}}=\frac{n}{c}\hat{n}\times\vec{\mathcal{E}}$. .16

- For harmonic fields it is customary to use complex notation (because of the spectral decomposition), so that $\vec{S}=\Re(\vec{E})\times\Re(\vec{H})$.
- In general for products of the type

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^*b+abe^{-2i\omega t}), \tag{34}$$

it is also customary to take time averages $\langle(\cdots)\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty(\cdots)dt$, and

$$\langle\Re(ae^{-i\omega t})\Re(be^{-i\omega t})\rangle = \frac{1}{2}\Re(a^*b). \tag{35}$$

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- We therefore have

$$\langle\vec{S}\rangle = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\mathcal{E}|^2\hat{n}. \tag{36}$$

And similarly,

$$\langle u\rangle = \frac{1}{4}(\epsilon\vec{E}\cdot\vec{E}^* + \frac{1}{\mu}\vec{B}\cdot\vec{B}^*) = \frac{\epsilon}{2}|\mathcal{E}|^2. \tag{37}$$

- Finally, $\langle\vec{S}\rangle = v_\phi u\hat{n}$. .18

2.2 Connection with radiative transfer

- We can now see that the concept of rays associated to the radiative transfer equation, which describes the transport of radiation in a straight line, is connected to the idea of a plane monochromatic wave with direction of propagation $\vec{k}$.
- For a plane wave then $\langle\vec{S}\rangle = v_\phi u\hat{k}$ is the flux of energy in direction $\hat{k}$.
- In radiative transfer notation, the flux density in direction $k_\circ$ would be

$$F_\nu(\vec{x}) = \int d\Omega I_\nu(\hat{k},\vec{x})\,\hat{k}\cdot\hat{k}_\circ. \tag{38}$$

- Therefore the specific intensity field for a monochromatic plane wave is

$$I_\nu(\hat{k}) = \|\vec{S}\|\,\delta(\hat{k}-\hat{k}_\circ). \tag{39}$$

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2.3 Polarization

- In summary, the electric field of a monochromatic wave can be decomposed in two linearly polarized waves,

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{40}$$

whose total describes, in general, an eliptically polarized wave.
- With a change of vectorial basis to $\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$, we can also decompose $\vec{E}$ in two circularly polarized waves,

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_+ E_+ + \hat{\epsilon}_- E_-)e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{41}$$

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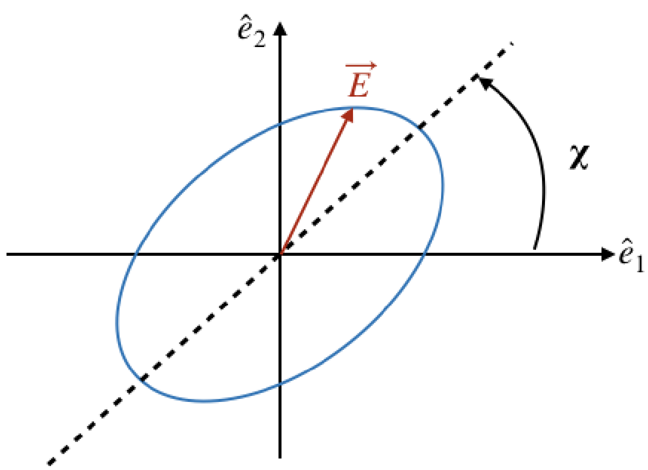
- With the notation

$$\begin{array}{rr} E_1 = \mathcal{E}_1 e^{i\phi_1}, & E_2 = \mathcal{E}_2 e^{i\phi_2}, \\ E_+ = \mathcal{E}_+ e^{i\phi_+}, & E_- = \mathcal{E}_- e^{i\phi_-}, \end{array}$$

we have

$$\begin{cases} \text{linear polarization} : \phi_2 - \phi_1 = 0. \\ \text{circular polarization} : |\phi_2 - \phi_1| = \frac{\pi}{2} \text{ and } \mathcal{E}_2 = \mathcal{E}_1. \\ \text{the general case is eliptical, with} : \tan(\chi) = \frac{\mathcal{E}_1}{\mathcal{E}_2}\frac{\cos(\phi_1)}{\cos(\phi_2)}. \end{cases}$$



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- It is customary to use the Stokes parameters to characterize the polarization state of **monochromatic light**:

$$\begin{array}{rll} I = & E_1E_1^* + E_2E_2^* = & \mathcal{E}_1^2 + \mathcal{E}_2^2, \\ Q = & E_1E_1^* - E_2E_2^* = & \mathcal{E}_1^2 - \mathcal{E}_2^2, \\ U = & E_1E_2^* - E_2E_1^* = & 2\mathcal{E}_1\mathcal{E}_2\cos(\phi_2-\phi_1), \\ V = & i(E_1E_2^* - E_2E_1^*) = & 2\mathcal{E}_1\mathcal{E}_2\sin(\phi_2-\phi_1). \end{array} \tag{42}$$

- We see that Stokes I (the total “radiance”) is $I \propto |\vec{S}|$, Q and U measure linear polarization, while V measure circular polarization. In order to make this obvious it is best to use mental experiments with polarizors that select specific types of polarization (see class).
- For a strictly monochromatic wave, it follows that

$$I^2 = Q^2 + U^2 + V^2. \tag{43}$$

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2.4 Quasi-monochromatic waves

- In order to obtain $\vec{E}(\vec{x}, \omega)$, we need to know $\vec{E}$(t) for all t, since

$$\vec{E}(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{x}, t) e^{i\omega t} dt. \tag{44}$$

- So in practice, we treat E_1 and E_2 as random variables, i.e. for a wave in vacuum, described by Eq. 40,

$$\vec{E}(\vec{x}, t) = (E_1(t)\hat{e}_1 + E_2(t)\hat{e}_2) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \tag{45}$$

  Alternatively we can also replace the time dependece with a probability density, which itself may depend on time.
- To fix ideas, let’s remember that $\Delta t \Delta \omega = 1$ for Gaussian spectra, where Δt is the ‘coherence time’, and $\Delta\omega$ is the ‘bandwidth’ of the quasi-monochromatic wave.

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- In order to measure the Stokes parameters, we need averages of the kind

$$\langle E_1 E_2^* \rangle = \lim_{T \to \infty} \frac{1}{T} \int dt E_1(t) E_2^*(t) dt. \tag{46}$$

- We therefore have

$$\begin{aligned} \langle Q^2 \rangle + \langle U^2 \rangle + \langle V^2 \rangle = \langle I^2 \rangle - \\ 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1 \mathcal{E}_2 e^{i(\phi_2 - \phi_1)} \rangle \langle \mathcal{E}_1 \mathcal{E}_2 e^{-i(\phi_2 - \phi_1)} \rangle \\ = \langle I^2 \rangle - \\ 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle), \end{aligned} \tag{47}$$

  and, by Shwartz’ inequality ($\langle ab \rangle \geq \langle a \rangle \langle b \rangle$),

$$I^2 \geq Q^2 + U^2 + V^2. \tag{48}$$

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- For a wave with a single and constant eliptical polarization state, then the equality holds in Eq. 48.
- On the other hand, for a completely unpolarized wave, $Q = U = V = 0$.
- The Stokes parameters are additive. Proof: consider a sum of N different waves

$$\vec{E} = \sum_{k=1}^{N} \vec{E}^k = \sum (\hat{\epsilon}_1 E_1^k + \hat{\epsilon}_2 E_2^k) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \tag{49}$$

Because each $E_i^k(t)$ is statistically independent, $\langle E_i^k E_j^{l*} \rangle = \delta_{kl} \langle E_i^k E_j^{k*} \rangle$, and

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \sum_k \begin{pmatrix} I_k \\ Q_k \\ U_k \\ V_k \end{pmatrix}. \tag{50}$$

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- We can therefore decompose an arbitrary set of Stokes parameters in

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \overbrace{\begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\text{unpol}} + \overbrace{\begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}}^{\text{pol}}. \tag{51}$$

- The first term 'unpol' is completely unpolarized since $Q = U = V = 0$, while the second term 'pol' is completely polarized since it satisfies $I^2 = Q^2 + U^2 + V^2$ (Eq. 43).

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- The total polarized intensity of a wave train is thus be $I^{\text{pol}} = \sqrt{Q^2 + U^2 + V^2}$.
- We define the polarization fraction as

$$\Pi = \frac{I^{\text{pol}}}{I}. \tag{52}$$

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3 Wave propagation in a medium

3.1 Constitutive equations

- Each monochromatic component of the field $\vec{E}, \vec{B}$ must fulfill the following constitutive relations:
$$\begin{array}{rcl} \vec{P} = \epsilon_\circ \chi \vec{E} & \longrightarrow & \vec{P}(\omega) = \epsilon_\circ \chi(\omega) \vec{E}(\omega), \\ \vec{B} = \mu \vec{H} & \longrightarrow & \vec{B}(\omega) = \mu(\omega) \vec{H}(\omega), \\ \vec{J} = \sigma \vec{E} & \longrightarrow & \vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega), \end{array} \tag{53}$$
in which we have added Ohm's law.
- We note that $\chi(-\omega) = \chi^*(\omega)$, so that $\chi(t) = \frac{1}{2\pi} \int d\omega \chi(\omega) \exp(-i\omega t)$ be real (and similarly for μ and σ).

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- The Fourier convolution theorem states that if $X(\omega) = Y(\omega) Z(\omega)$, then
$$X(t) = \int_{-\infty}^{\infty} Y(t - t') Z(t') dt', \tag{54}$$
where $Y(t) = \frac{1}{2\pi} \int d\omega Y(\omega) \exp(-i\omega t)$, etc..
- Applying the convolution theorem to χ (for example),
$$P(t) = \int_{-\infty}^{\infty} G(t - t') E(t') dt', \text{ with}$$
$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_\circ \chi(\omega) e^{-i\omega t} d\omega. \tag{55}$$

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- We see that $P(t)$ depends on the history of $\vec{E}(t')$, which bears physical sense only in the past, for $t' < t$, so $G(t) = 0$ if $t < 0$. We will use this property in the next section.
- This time we write the monochromatic wave as
$$\vec{E}(t) = \vec{A} \cos(\omega_\circ t) + \vec{B} \sin(\omega_\circ t) = \Re(\vec{E}_c(t)), \tag{56}$$
with $\vec{E}_c = (\vec{A} - i\vec{B})(\cos(\omega_\circ t) + i \sin(\omega_\circ t))$.
- In the Fourier plane,
$$E(\omega) = \pi \left[(A + iB) \delta(\omega - \omega_\circ) + (A - iB) \delta(\omega + \omega_\circ) \right]. \tag{57}$$
- We can evaluate
$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_\circ \chi(\omega) E(\omega) e^{-i\omega t} dt$$
$$= \Re \left[\frac{\epsilon_\circ}{2} (A - iB) \chi(\omega_\circ) e^{-i\omega_\circ t} \right] = \Re[P_c(t)], \tag{58}$$
using $\chi(-\omega) = \chi^*(\omega)$, and where $P_c = \epsilon_\circ \chi(\omega_\circ) E_c(t)$.

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- With the spectral decomposition of the constitutive relations we can rewrite the Maxwell equations in their harmonic versions. In the absence of free charges,

$$\begin{array}{ll} \vec{\nabla}\cdot\vec{E}(\omega)=0, & \vec{\nabla}\times\vec{E}(\omega)=-i\omega\mu(\omega)\vec{H}(\omega), \\ \vec{\nabla}\cdot\vec{H}(\omega)=0, & \vec{\nabla}\times\vec{H}(\omega)=-i\omega\epsilon(\omega)\vec{E}(\omega), \end{array} \tag{59}$$

where

$$\epsilon(\omega)=\epsilon_\circ(1+\chi(\omega))+i\frac{\sigma(\omega)}{\omega}. \tag{60}$$

- Note that both susceptibility and conductivity contribute to the imaginary part of ϵ:

$$\Im(\epsilon)=\epsilon_\circ\Im(\chi)+\Re(\sigma/\omega). \tag{61}$$

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3.2 Kramers-Kronig relations

- From physical considerations we can anticipate that the induced $P(t)$ depends on the history of the applied field, or

$$\vec{P}(t)=\int_{-\infty}^{\infty}G(t,t')\vec{E}(t')dt' \tag{62}$$

(note difference with Eq. 55).
- Let's assume that $\vec{E}=\delta(t-t_\circ)\vec{E}_\circ$. Then $\vec{P}(t)=G(t,t_\circ)\vec{E}_\circ$, and G is the polarization resulting from a delta-unitary electric field.
- If the properties of the medium do not change in time, $G(t,t_\circ)=G(t-t_\circ)$, and we recover Eq. 55.
- Causality requires that $G(\tau)=0$ if $\tau<0$, so

$$\epsilon_\circ\chi(\omega)=\int_0^{\infty}dtG(t)e^{i\omega t}. \tag{63}$$

Proof: apply Eq. 55 to an harmonic field $\vec{E}=\vec{E}_\circ\exp(i\omega t)$.

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- We extend Eq. 63 to the complex plane with $\tilde{\omega}=\omega_R+i\omega_I$, where $\omega_I>0$.

$$\epsilon_\circ\chi(\tilde{\omega})=\int_0^{\infty}dtG(t)e^{i\tilde{\omega}t}. \tag{64}$$

- If $\int_0^\infty|G(t)|dt$ converges, so does $\int_0^\infty G(t)e^{i\tilde{\omega}t}dt$, and $\chi(\tilde{\omega})$ is analytical in the superior $\mathbb{C}$ plane ($\omega_I>0$).
- Therefore $\chi(\tilde{\omega})/(\tilde{\omega}-\omega)$ is analítical except in the pole $\tilde{\omega}=\omega$, where ω is a point in the real axis.

- We can apply the Kramers-Kronig theorem (proof: see Bohren & Huffman, Sec. 2.3.2), which gives

$$i\pi\chi(\omega) = P\int_{-\infty}^{\infty}\frac{\chi(\Omega)}{\Omega+\omega}d\Omega, \tag{65}$$

where P indicates Cauchy's 'principal value'

$$P\int_{-\infty}^{\infty}\frac{\chi(\Omega)}{\Omega+\omega}d\Omega = \lim_{a\to 0}\left(\int_{-\infty}^{\omega-a}\frac{\chi(\Omega)}{\Omega+\omega}d\Omega + \int_{\omega+a}^{\infty}\frac{\chi(\Omega)}{\Omega+\omega}d\Omega\right). \tag{66}$$

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- Using that $\chi^*(\Omega) = \chi(-\Omega)$ we can restrict the integration to $\Omega > 0$, and use $\chi = \chi_R + i\chi_I$ to rewrite Eq. 66:

$$\chi_R(\omega) = \frac{2}{\pi}P\int_0^{\infty}\frac{\Omega\chi_I(\Omega)}{\Omega^2-\omega^2}d\Omega, \tag{67}$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi}P\int_0^{\infty}\frac{\chi_R(\Omega)}{\Omega^2-\omega^2}d\Omega. \tag{68}$$

- Similar relationships exists for μ y σ.

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3.3 Monochromatic waves

- We now extend the monochromatic waves to homogeneous media. We inject

$$\vec{E}_c = \vec{E}_\circ e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \text{ and } \vec{H}_c = \vec{H}_\circ e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{69}$$

into Maxwell's equations.

- Allowing for $\vec{k} \in \mathbb{C}$, $\vec{k} = \underbrace{(k_R + ik_I)}_{k}\hat{e}$,

$$\vec{E}_c = \vec{E}_\circ e^{-\vec{k}_I\cdot\vec{x}}e^{i(\vec{k}_R\cdot\vec{x}-\omega t)}. \tag{70}$$

- The harmonic Maxwell equations (Eqs. 59) yield:

$$\begin{array}{cc} \vec{k}\cdot\vec{E}_\circ = 0 & \vec{k}\cdot\vec{H}_\circ(\omega) = 0 \\ \vec{k}\times\vec{E}_\circ = \omega\mu\vec{H}_\circ, & \vec{k}\times\vec{H}_\circ = -\omega\epsilon\vec{E}_\circ. \end{array} \tag{71}$$

- And with $\vec{k}\cdot\vec{k} = \omega^2\epsilon\mu$,

$$k_R^2 - k_I^2 + 2i\vec{k}_I\cdot\vec{k}_R = \omega^2\epsilon\mu \ \text{(tarea)}. \tag{72}$$

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- For a homogeneous wave (no free charges),

$$\vec{k} = \underbrace{(k_R + ik_I)}_{k}\hat{e},$$

and $k = \omega N/c$, where N is the complex refractive index,

$$N = c\sqrt{\epsilon\mu} = \sqrt{\frac{\epsilon\mu}{\epsilon_\circ\mu_\circ}}.$$

- We set $N = n + i\kappa$, where n and κ are both $\in \mathbb{R}^+$.
- Eq. 70 gives:

$$\vec{E}_c = \vec{E}_\circ e^{-\frac{2\pi}{\lambda}\kappa z} e^{i(\frac{2\pi n z}{\lambda} - i\omega t)}. \tag{73}$$

- $\Rightarrow$ the imaginary part of N corresponds to absorption.

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- We can apply the Kramers-Kronig relations to $(N(\omega)-1)$ (the -1 is motivated by $\lim_{\omega\to\infty} N(\omega) = 1$):

$$\begin{array}{rcl} n(\omega) - 1 & = & \frac{2}{\pi} P \int_0^\infty \frac{\Omega\kappa(\Omega)}{\Omega^2-\omega^2} d\Omega \\ \kappa(\omega) & = & -\frac{2\omega}{\pi} P \int_0^\infty \frac{n(\Omega)}{\Omega^2-\omega^2} d\Omega \end{array} \tag{74}$$

- We see that the absorption in a medium is also related to the real refractive index.

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