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Part III
Radiation

Contents

1 Green function for the wave equation

- In order to determine $\vec{A}(\vec{x},t)$ and $\Phi(\vec{x},t)$, we need to solve the wave equation with source terms. For a generic field $\Psi(\vec{x},t)$,

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial\Psi}{\partial t^2} = -4\pi f(\vec{x},t). \tag{1}$$

- It is convenient to use the Fourier time-domain,

$$\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega, \tag{2}$$

whose inverse is

$$\psi(\vec{x},\omega) = \int_{-\infty}^{+\infty}\psi(\vec{x},t)e^{i\omega t}dt. \tag{3}$$

- Injecting Ec. 2 in Ec. 1, we reach the Helmholtz equation:

$$(\nabla^2 + k^2)\Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega). \quad (4)$$

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- The Helmholtz equation Eq. 4 is very similar to the Poisson equation, and we can anticipate the use of similar machinery in its solution. The Green function satisfies

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}, \vec{x}'). \quad (5)$$

- Changing coordinates to a system with origin in $\vec{x}'$, we see that Eq. 5 has spherical symmetry, and $G_k(\vec{x}, \vec{x}') = G_k(R)$, with $R = |\vec{R}|$ and $\vec{R} = \vec{x} - \vec{x}'$
- Eq. 5 can thus be written as

$$\frac{1}{R}\frac{d^2}{dR^2}(R\,G_k) + k^2 G_k = -4\pi\delta(\vec{R}). \quad (6)$$

- If $R \neq 0$, the solution to Eq. 6 is $RG_k = Ae^{ikR} + Be^{-ikR}$, where the constants A y B do not depend on k. In order to dertermine these constants, we use the case $k = 0$, i.e. Poisson, whose solution is $G_{k=0}(R) = 1/R$, $\longrightarrow A + B = 1$.
- Thus the general solution to Eq. 6 is

$$G_k(R) = AG_k^+(R) + BG_k^-(R), \text{ with } G_k^{\pm} = \frac{e^{\pm ikR}}{R} \quad \text{and } A + B = 1. \quad (7)$$

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- The A and B values depend on the initial conditions, i.e. on the boundary conditions in time. To see this, we return to the time domain and we generalize Eq. 5:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right) G^{\pm}(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \quad (8)$$

- Now, returning to the frequency domain ω, we generalize Eq. 5 to

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}'; t') = -4\pi\delta(\vec{x}, \vec{x}')e^{i\omega t'}, \quad (9)$$

  with solution $G_k^{\pm}(R)e^{i\omega t'}$.
- To return once more to the time domain, we use Eq. 2, and

$$G^{\pm}(R; t, t') = G^{\pm}(R, \tau) = \frac{1}{2\pi}\int \frac{e^{\pm ikR - i\omega\tau}}{R} d\omega,$$

  where $\tau = t - t'$.

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- For a non-dispersive medium (one with $\omega/k = c$), we reach

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right]\right)}{|\vec{x}-\vec{x}'|} \tag{10}$$

- We apply this Green function to write the generic solutions of Ec. 1:

$$\Psi^{\pm}(\vec{x},t) = \int d^3x'dt'G^{\pm}(\vec{x},t;\vec{x}',t')f(\vec{x}',t'). \tag{11}$$

- The $^+$ case corresponds to the retarded solution, with an entry or an initial condition ψ_{in} (valid before the sources f are activated, at $t=0$, so $f(\vec{x},t)=0$ if $t<0$):

$$\Psi^{+}(\vec{x},t) = \Psi_{\text{in}}(\vec{x},t) + \int d^3x'dt'G^{+}(\vec{x},t;\vec{x}',t')f(\vec{x}',t'), \tag{12}$$

where we see that if $t<0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t - \frac{|\vec{x}-\vec{x}'|}{c}\right] > 0$. Hence if $t<0$, $\int d^3x'dt'G^{+}(\vec{x},t;\vec{x}',t')f(\vec{x}',t') = 0$, and $\Psi(\vec{x},t) = \Psi_{\text{in}}(\vec{x},t)$.

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- Instead the $^-$ case corresponds to the anticipated solution, with an exit condition Ψ_{out} (after the sources f are deactivated, at $t=0$, so $f(\vec{x},t)=0$ if $t>0$)),

$$\Psi^{-}(\vec{x},t) = \Psi_{\text{out}}(\vec{x},t) + \int d^3x'dt'G^{-}(\vec{x},t;\vec{x}',t')f(\vec{x}',t'), \tag{13}$$

where we see that if $t>0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t + \frac{|\vec{x}-\vec{x}'|}{c}\right] < 0$. Hence if $t>0$, $\int d^3x'dt'G^{-}(\vec{x},t;\vec{x}',t')f(\vec{x}',t') = 0$, and $\Psi(\vec{x},t) = \Psi_{\text{out}}(\vec{x},t)$.
- In general we use the retarded solution Ψ^+.

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2 Retarded Potentials

2.1 Application of the Green function to the electrodynamic potentials

- We normally use the retarded solution, with initial condition $\Psi_{\text{in}} = 0$, or, in compact notation,

$$\Psi(\vec{x},t) = \int d^3x' \frac{[f(\vec{x}',t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|}, \tag{14}$$

where $[(\cdots)]_{\text{re}}$ means to evaluate in $t' = t - |\vec{x}-\vec{x}'|/c$.

- Applying to the electrodynamic potentials,

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_\circ}\int d^3x' \frac{[\rho(\vec{x}',t')]_{\rm ret}}{|\vec{x}-\vec{x}'|}, \quad (15)$$

$$\vec{A}(\vec{x},t) = \frac{\mu_\circ}{4\pi}\int d^3x' \frac{[\vec{J}(\vec{x}',t')]_{\rm ret}}{|\vec{x}-\vec{x}'|}. \quad (16)$$

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2.2 Retarded electromagnetic field

- In order to calculate $\vec{E}$ and $\vec{B}$, we use $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$.
- Alternatively, we can use the Maxwell equations to reach:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\epsilon_\circ}\left(-\vec{\nabla}\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right), \quad (17)$$

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\mu_\circ\vec{\nabla}\times\vec{J}. \quad (18)$$

- Using the Green function, we get

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_\circ}\int d^3x' \frac{1}{R}\left[-\vec{\nabla}'\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t'}\right]_{\rm ret}, \quad (19)$$

$$\vec{B}(\vec{x},t) = \frac{\mu_\circ}{4\pi}\int d^3x' \frac{1}{R}\left[\vec{\nabla}'\times\vec{J}\right]_{\rm ret}. \quad (20)$$

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- The expressions for the retarded fields Eqs. 19 and 21 can also be written in a form that connects directly with the static expressions (tarea):

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_\circ}\int d^3x' \left\{\frac{\hat{R}}{R^2}\left[\rho(\vec{x}',t')\right]_{\rm ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\vec{x}',t')}{\partial t}\right]_{\rm ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{J}}{\partial t}\right]_{\rm ret}\right\} \quad (21)$$

$$\vec{B}(\vec{x},t) = \frac{\mu_\circ}{4\pi}\int d^3x' \left\{\left[\vec{J}(\vec{x}',t')\right]_{\rm ret}\times\frac{\hat{R}}{R^2} + \left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t}\right]_{\rm ret}\times\frac{\hat{R}}{cR}\right\} \quad (22)$$

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3 Multipolar Radiation

3.1 Wave zone

- We now consider harmonic sources (the general case can be obtained by superposition of such sources):

$$\begin{aligned} \rho(\vec{x}, t) &= \rho(\vec{x}) e^{-i\omega t}, \\ \vec{J}(\vec{x}, t) &= \vec{J}(\vec{x}) e^{-i\omega t}. \end{aligned} \tag{23}$$

- We saw that in the presence of sources, the field $\vec{A}(\vec{x}, t)$ generated in vacuum, and without spatial boundaries, is

$$\vec{A}(\vec{x}, t) = \frac{\mu_\circ}{4\pi} \int d^3x' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta\left(t' + \frac{|\vec{x} - \vec{x}'|}{c} - t\right).$$

- For harmonic sources,

$$\vec{A}(\vec{x}, t) = e^{-i\omega t} \frac{\mu_\circ}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}, \tag{24}$$

  with $k = \omega/c$ .11

- We then obtain $\vec{H}$ and $\vec{E}$ with

$$\vec{H} = \frac{1}{\mu_\circ} \vec{\nabla} \times \vec{A}, \tag{25}$$

  and Faraday's law,

$$\vec{E} = \frac{i}{k} \sqrt{\frac{\mu_\circ}{\epsilon_\circ}} \vec{\nabla} \times \vec{H}. \tag{26}$$

- We now consider sources confined inside a region whose maximum extension is d, and that contains the origin. If $d \ll \lambda$, there are 3 regions of interest:
  - The near zone, with $d < r \ll \lambda$, where $e^{ik|\vec{x} - \vec{x}'|} \sim 1$ and we recover the static potentials except for harmonic oscilation, $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x}) e^{-i\omega t}$.
  - The intermediate zone with $d \ll r \sim \lambda$.
  - The far zone, with $\lambda \ll r$. .12

- In the far zone, with $\lambda \ll r$, $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$, where $\hat{n} = \vec{x}/r \Rightarrow$

$$\lim_{kr \to \infty} \vec{A}(\vec{x}) = \frac{\mu_\circ}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') e^{-ik\hat{n} \cdot \vec{x}'} d^3x'. \tag{27}$$

- We see that $\vec{A}(x, t) = \vec{A}(\vec{x}) e^{-i\omega t}$ represents a spherical wave travelling outwards.
- In addition, (tarea) using Eqs 25 and 26 we also see that $\vec{E}$ and $\vec{H}$ also form spherical transverse waves (orthogonal to $\hat{n}$).
- The far zone thus corresponds to the *radiation zone*, also called *wave zone*. .13

3.2 Dipolar radiation

- We now use $d \ll \lambda$ to simplify $\vec{A}$ in the wave zone. The integrand in Eq. 27 can be expanded in powers of $-ik\hat{n}\cdot\vec{x}'$, using

$$e^{-ik\hat{n}\cdot\vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!}(\hat{n}\cdot\vec{x}')^n.$$

- Therefore,

$$\vec{A}(\vec{x}) = \frac{\mu_\circ}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(\hat{n}\cdot\vec{x}')^n d^3x'. \tag{28}$$

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- For $n=0$, which is the dominant term in the expansion in $k\hat{n}\cdot\vec{x}'$, we get:

$$\vec{A}(\vec{x}) = \frac{\mu_\circ}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')d^3x'. \tag{29}$$

- Using the continuity equation, $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$, we have $i\omega\rho = \vec{\nabla}\cdot\vec{J}$.
- Therefore (tarea):

$$\int \vec{J}(\vec{x}')d^3x' = -\int \vec{x}'(\vec{\nabla}'\cdot\vec{J})d^3x' = -i\omega\int \vec{x}'\rho(\vec{x}')d^3x'. \tag{30}$$

- Finally,

$$\vec{A}(\vec{x}) = \frac{-i\mu_\circ\omega}{4\pi}\vec{p}\frac{e^{ikr}}{r}, \quad \text{with} \quad \underbrace{\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x'}_{\text{elecric dipole}}. \tag{31}$$

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- We now calculate the $\vec{E}$ and $\vec{H}$ fields:

$$\begin{aligned} \vec{H} &= \tfrac{ck^2}{4\pi}(\hat{n}\times\vec{p})\tfrac{e^{ikr}}{r}, \\ \vec{E} &= \sqrt{\mu_\circ\epsilon_\circ}\vec{H}\times\hat{n}, \end{aligned} \tag{32}$$

where we see that electric dipole radiation is linearly polarized.
- The power emitted in direction $\hat{n}$ can be written with $dP = r^2 d\Omega\hat{n}\cdot\vec{S} \ \Rightarrow$,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2}\Re\left[r^2\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right], \\ &= \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_\circ}{\epsilon_\circ}}k^4|\vec{p}|^2\sin^2(\theta). \end{aligned} \tag{33}$$

- The total power is

$$P = \frac{c^2k^4}{12\pi}|\vec{p}|^2. \tag{34}$$

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3.3 Magnetic dipole and electric quadrupole radiation

- The next term in the expansion of $e^{-ik\hat{n}\cdot\vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!}(\hat{n}\cdot\vec{x}')^n$ corresponds to $n = 1$.
- Eq. 28 gives:

$$\vec{A}(\vec{x}) = \frac{\mu_\circ}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int \vec{J}(\vec{x}')(\hat{n}\cdot\vec{x}')d^3x'. \tag{35}$$

- This term originates *magnetic dipole* and *electric quadrupole* contributions. To see this, we separate the integrand:

$$\vec{J}(\vec{x}')(\hat{n}\cdot\vec{x}') = \underbrace{\frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right]}_{A} + \underbrace{\frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}}_{B}. \tag{36}$$

- We first consider the contribution of part B and identify the magnetization $\vec{\mathcal{M}}$,

$$\vec{\mathcal{M}} = \frac{1}{2}(\vec{x}\times\vec{J}). \tag{37}$$

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- Then, for the B part,

$$\vec{A}(\vec{x}) = \frac{ik\mu_\circ}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}(1 - \frac{1}{ikr}), \quad \text{with} \tag{38}$$

$$\vec{m} = \int \vec{\mathcal{M}} d^3x. \tag{39}$$

- In the radiation zone, $kr \gg 1$, we obtain:

$$\vec{A}(\vec{x}) = \frac{ik\mu_\circ}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}, \tag{40}$$

$$\vec{E}(\vec{x}) = -\frac{k^2}{4\pi}\sqrt{\frac{\mu_\circ}{\epsilon_\circ}}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}, \tag{41}$$

$$\vec{H}(\vec{x}) = -\sqrt{\frac{\epsilon_\circ}{\mu_\circ}}(\vec{E}\times\hat{n}). \tag{42}$$

- This contribution is called magnetic dipole radiation.

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- For the A part in the contribution from $n = 1$, after standard handling we get:

$$A = \frac{i\omega}{2}\int \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')d^3x', \tag{43}$$

which represents order 2 moments of $\rho(\vec{x}')$, i.e. an electric quadrupole contribution, which we will not develop.

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4 Radiation from a single charge

4.1 Liénard-Wiechert potentials

- Consider a particle with charge q and trajectory $\vec{r}(t)$, with velocity $\vec{u}(t)$.
- We can apply the retarded solution with source terms $\rho(\vec{x},t) = q\delta(\vec{x} - \vec{r}(t))$, and $\vec{j}(\vec{x},t) = \rho(\vec{x},t)u(t)$, and thus obtain the resulting potentials (tarea):

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_\circ}\left[\frac{q}{\left(1 - \hat{n}(t')\cdot\vec{\beta}(t')\right)R(t')}\right]_{\text{ret}}, \tag{44}$$

$$\vec{A}(\vec{x},t) = \frac{\mu_\circ}{4\pi}\left[\frac{q\vec{u}}{\left(1 - \hat{n}(t')\cdot\vec{\beta}(t')\right)R(t')}\right]_{\text{ret}}, \tag{45}$$

where $\vec{R}(t') = \vec{x} - \vec{r}(t')$, $R = |\vec{R}|$, $\hat{n}(t') = \frac{\vec{R}}{R}$, and $\vec{\beta}(t') = \frac{\vec{u}(t')}{c}$. These are the Liénard-Wiechert (L.-W.) potentials.

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- We can also calculate the corresponding electromagnetic field (tarea):

$$\vec{E}(\vec{x},t) = \overbrace{\left[\frac{q}{4\pi\epsilon_\circ}\frac{(1-\beta^2)\left(\hat{n}-\vec{\beta}\right)}{R^2(1-\hat{n}\cdot\vec{\beta})^3}\right]_{\text{ret}}}^{E_{\text{vel}}} + \underbrace{\left[\frac{q}{4\pi\epsilon_\circ}\frac{\left(\hat{n}\times\left(\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)\right)}{cR\left(1-\hat{n}\cdot\vec{\beta}\right)^3}\right]_{\text{ret}}}_{E_{\text{rad}}} \tag{46}$$

with

$$\vec{B}(\vec{x},t) = \frac{1}{c}\hat{n}\times\vec{E}(\vec{x},t). \tag{47}$$

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- We see that far away from the particle, the term labelled E_{rad} will eventually dominate. In fact, for a Fourier component, or for harmonic motion with $\vec{r}(t) \propto \exp(i\omega t)$, we find that (tarea)

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} = \beta\frac{R}{\lambda}, \tag{48}$$

and we see that the radiation term dominates if $R \gg \lambda/\beta$, sometimes also called the *far zone*.

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4.2 Larmor Formula

- In the far zone, with $R \to \infty$, and in the Galilean limit ($\beta \ll 1$), the power radiated per unit solid angle is (tarea):

$$\frac{dP}{d\Omega} = \frac{\mu_\circ}{16\pi^2} q^2\, a^2 \sin^2(\theta), \tag{49}$$

in which $a = |\dot{\vec{u}}|$ and θ is the angle between $\hat{n}$ and $\vec{a}$. This is the Larmor formula.
- By applying Eq. 49 to the case of an harmonically oscillating charge with dipole $\vec{p} = q\vec{r}_\circ \exp(i\omega t)$, we can recover Eq. 33 (tarea).
- We therefore conclude that the *wave zone* as defined in Sec. 3.1 matches the Galilean limit (see the discussion on the dipole approximation in Sec. 3.3 of Rybicki & Lightman).

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4.3 Radiation reaction

- Let's consider a periodic sistem, such that its mechanical state is identical between times t_1 and t_2.
- Still in the Galilean limit, the total energy radiated by the charge between t_1 and t_2 is

$$W = \frac{\mu_\circ q^2}{6\pi c} \int_{t1}^{t2} a^2 dt. \tag{50}$$

- Energy conservation requires the existence of 'radiation reaction' force $F_{\rm rad}$, such that W is extracted from the particle's kinetic energy,

$$\int_{t1}^{t2} dt \vec{F}_{\rm rad}.\vec{u} = -W. \tag{51}$$

- One expression for the radiation reaction is the Abraham-Lorentz formula,

$$\vec{F}_{\rm rad} = \frac{\mu_\circ q^2}{6\pi c} \dot{\vec{a}}. \tag{52}$$

We can confirm that Eq. 52 indeeds fullfills Eq. 51 (tarea).

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5 Non-relativistic applications

- Thomson scattering (Rybicki & Lightman Sec. 3.4)
- Harmonically bound particles (Rybicki & Lightman Sec. 3.6)

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