.1

Part IV

Free-free, Synchrotron and Compton Scattering

Contents

1 Relativity

1.1 Quadrivectors

In this section we switch to CGS units, better adapted to describe the symmetry between $\vec{E}$ and $\vec{B}$ (bibliography: Rybicki & Lightman).

- We define $x^\mu = (ct, x, y, z)$ as the contravariant position quadrivector, whose norm is $s^2 = \eta_{\mu\nu} x^\mu x^\nu$ (using the implicit sum notation).
- We also introduce $x_\mu = (-ct, x, y, z)$ as the covariant position quadrivector.
- $x_\mu = \eta_{\mu\nu} x^\nu$, with

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

- With $\eta^{\mu\nu} \equiv \eta_{\mu\nu}$, we have $x^\mu = \eta^{\mu\nu} x_\nu$.
- Note that $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$. .3

- We change reference system from $\mathcal{S}$ to $\mathcal{S}'$, in uniform translation with velocity v towards $\hat{x}$ relative to $\mathcal{S}$.
- A contravariant 4V transforms as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \text{ where } \Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

with $\beta = v/c$, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

- For a covariant 4V (tarea),

$$x'_\mu = \tilde{\Lambda}_\mu^{\ \nu} x_\nu, \quad \text{with } \tilde{\Lambda}_\mu^{\ \nu} = \eta_{\mu\tau} \Lambda^\tau_{\ \sigma} \eta^{\sigma\nu}. \tag{3}$$

- $\tilde{\Lambda}_\mu^{\ \nu}$ is the inverse of $\Lambda^\mu_{\ \nu}$:

$$\Lambda^\sigma_{\ \nu} \tilde{\Lambda}_\sigma^{\ \mu} = \delta^\mu_{\ \nu}, \quad \text{and } \tilde{\Lambda}^\alpha_\mu x'^\mu = x^\alpha. \tag{4}$$

.4

- The product of two 4Vs A^μ and B^μ is Lorentz invariant:

$$A^\mu B_\mu = A'^\mu B'_\mu. \tag{5}$$

- We also have the velocity 4V, $U^\mu \equiv \frac{dx^\mu}{d\tau}$, in which $d\tau$ is the relativistic interval. (i.e. of proper time) between x^μ and $x^\mu + dx^\mu$.
- In components (tarea), $U^\mu = \gamma_u(c, \vec{u})$, with $\gamma_u = 1/\sqrt{1 - u^2/c^2}$.
- If we change to $U' = \Lambda^\mu_{\ \nu} U^\nu$,

$$\gamma_{u'} = \gamma\gamma_u(1 - \frac{uv}{c^2}\cos(\theta)), \quad \text{with } \theta = \angle(\vec{u}, \vec{v}). \tag{6}$$

- In the system that is bound to a particle with velocity $\vec{u}$, $U' = c(1, \vec{0})$, for a 4V A^μ,

$$A'^0 = -\frac{1}{c} U^\mu A_\mu = -\frac{1}{c} U'^\mu A'_\mu.. \tag{7}$$

.5

- We note that the phase of a plane wave must be Lorentz invariant because the simultaneous cancellation of $\vec{E}$ and $\vec{B}$ in one system implies their cancellation in any other system.
- Let's introduce $k^\mu = (\omega/c, \vec{k})$:

$$k^\mu x_\mu = \vec{k} \cdot \vec{x} - \omega t = \text{ invariant } \Rightarrow \ k^\mu \text{ is 4V}. \tag{8}$$

- We can use Eq. 7 to deduce the relativistic Doppler effect (tarea)

$$ck'^0 = \omega' = -U^\mu k_\mu = \omega\gamma(1 - \frac{v}{c}\cos(\theta)). \tag{9}$$

.6

- The gradient operator is another example of 4V. If λ is a scalar invariant, then

$$\lambda_{,\mu} \equiv \frac{\partial \lambda}{\partial x^{\mu}} \text{ is a covariant 4V, and} \tag{10}$$

$$\lambda^{,\mu} \equiv \frac{\partial \lambda}{\partial x_{\mu}} \text{ is a contravariant 4V.} \tag{11}$$

- Proof: from $x^{\nu} = \tilde{\Lambda}_{\mu}^{\ \nu} x'^{\mu}$, we have that $\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \tilde{\Lambda}_{\mu}^{\ \nu}$, and since $\lambda' = \lambda$,

$$\lambda'_{,\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \lambda}{\partial x^{\nu}}. \tag{12}$$

- We extend the properties of 4Vs to tensors in general: a tensor of orden n transforms as the product of n 4Vs.
- For example,

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \sigma} \Lambda^{\nu}_{\ \tau} T^{\sigma\tau},$$
$$T'^{\mu}_{\ \nu} = \Lambda^{\mu}_{\ \sigma} \tilde{\Lambda}_{\nu}^{\ \tau} T^{\sigma}_{\ \tau}.$$

.7

1.2 Covariance in electrodynamics

- Charge conservation, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, can be written as

$$J^{\mu}_{\ ,\mu} = 0, \text{ using the four-current } J^{\mu} = (\rho\, c, \vec{J}). \tag{13}$$

- In the Lorentz Gauge , and using CGS units,

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J} = \partial_{\alpha} \partial^{\alpha} \vec{A}, \tag{14}$$

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho = \partial_{\alpha} \partial^{\alpha} \Phi. \tag{15}$$

- With $A^{\mu} = (\Phi, \vec{A})$,

$$A^{\beta,\alpha}_{,\alpha} = -\frac{4\pi}{c} J^{\beta}, \text{ in which } A^{\beta,\alpha}_{,\alpha} = \frac{\partial^2}{\partial x_{\alpha} x^{\alpha}} A^{\beta}. \tag{16}$$

- The Lorentz gauge $\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0$ can be written simply as $A^{\alpha}_{\ ,\alpha} = 0$.

.8

- In order to write the Maxwell equations in their covariant form, we introduce the field tensor

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{17}$$

- With $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ (tarea): 2

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{18}$$

- The Maxwell equations $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and $\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$ can be written (tarea)

$$F_{\mu\nu}{}^{;\nu} = \frac{4\pi}{c} J_\mu. \tag{19}$$

- The 'internal' equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ are written (tarea)

$$F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0. \tag{20}$$

.9

- We use the covariance of $F_{\mu\nu}$ to infer the transformation laws for the fields $\vec{E}$ y $\vec{B}$:

$$F'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta F_{\alpha\beta}. \tag{21}$$

- In terms of components we get (tarea):

$$\begin{array}{ll} E'_{\parallel} = E_{\parallel}, & B'_{\parallel} = B_{\parallel}, \\ E'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}), & B'_{\perp} = \gamma(\vec{B}_{\perp} + \vec{\beta} \times \vec{E}). \end{array} \tag{22}$$

- We see that $\vec{E}$ and $\vec{B}$ get mixed up, and if $\vec{B} = 0$ in $\mathcal{S}$, then when changing to $\mathcal{S}'$ we have $\vec{B}' \neq 0$.

.10

- In order to extend the Lorentz force, we introduce the momentum quadrivector (four-momentum) $P^\mu = m_\circ U^\beta$, where $m_\circ$ is the rest mass. We write $P^\mu = (E/c, \vec{P})$, in which E is the total energy of the particle (which is $E = m_\circ c^2$ at rest).
- The acceleration 4V (four-acceleration) is

$$a^\mu = \frac{dU^\mu}{d\tau}, \tag{23}$$

and in order to recover Newton's 2nd law in the non-relativistic limit, the four-force must be

$$F^\mu = m_\circ a^\mu = \frac{dP^\mu}{d\tau}. \tag{24}$$

- We write the 4-Lorentz force with

$$F^{\mu} = \frac{q}{c} F^{\mu}{}_{\nu} U^{\nu}. \tag{25}$$

- in components, (tarea) $\vec{F} = q(\frac{\vec{v}}{c} \times \vec{B}) + q\vec{E}$.

.11