# Radiative Processes

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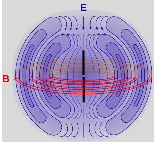
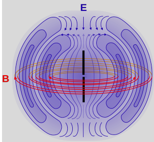
# Part V

## Free-free, Synchrotron and Compton Scattering



# Outline

1. **Relativity**
   - Quadrivectors
   - Covariance in electrodynamics

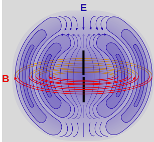


## Outline

1 **Relativity**

Quadrivectors

Covariance in electrodynamics



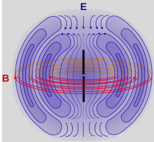
# 1.1- Quadrivectors

In this section we switch to CGS units, better adapted to describe the symmetry between $\vec{E}$ and $\vec{B}$ (bibliography: Rybicki & Lightman).

- We define $x^\mu = (ct, x, y, z)$ as the contravariant position quadrivector, whose norm is $s^2 = \eta_{\mu\nu} x^\mu x^\nu$ (using the implicit sum notation).
- We also introduce $x_\mu = (-ct, x, y, z)$ as the covariant position quadrivector.
- $x_\mu = \eta_{\mu\nu} x^\nu$, with

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

- With $\eta^{\mu\nu} \equiv \eta_{\mu\nu}$, we have $x^\mu = \eta^{\mu\nu} x_\nu$.
- Note that $\eta^{\mu\sigma} \eta_{\sigma\nu} = \delta^\mu_\nu$.



# 1.1- Quadrivectors

- We change reference system from $\mathcal{S}$ to $\mathcal{S}'$, in uniform translation with velocity $v$ towards $\hat{x}$ relative to $\mathcal{S}$.
- A contravariant 4V transforms as

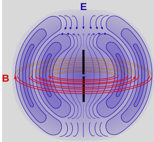
$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \text{ where } \Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

  with $\beta = v/c$, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.
- For a covariant 4V (tarea),

$$x'_{\mu} = \tilde{\Lambda}_{\mu}{}^{\nu} x_{\nu}, \quad \text{with } \tilde{\Lambda}_{\mu}{}^{\nu} = \eta_{\mu\tau} \Lambda^{\tau}{}_{\sigma} \eta^{\sigma\nu}. \tag{3}$$

- $\tilde{\Lambda}_{\mu}{}^{\nu}$ is the inverse of $\Lambda^{\mu}{}_{\nu}$:

$$\Lambda^{\sigma}{}_{\nu} \tilde{\Lambda}_{\sigma}{}^{\mu} = \delta^{\mu}_{\nu}, \quad \text{and } \tilde{\Lambda}^{\alpha}_{\mu} x'^{\mu} = x^{\alpha}. \tag{4}$$



## 1.1- Quadrivectors

- The product of two 4Vs $A^\mu$ and $B^\mu$ is Lorentz invariant:

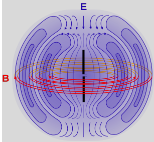
$$A^\mu B_\mu = A'^\mu B'_\mu . \tag{5}$$

- We also have the velocity 4V, $U^\mu \equiv \frac{dx^\mu}{d\tau}$, in which $d\tau$ is the relativistic interval. (i.e. of proper time) between $x^\mu$ and $x^\mu + dx^\mu$.
- In components (tarea), $U^\mu = \gamma_u(c, \vec{u})$, with $\gamma_u = 1/\sqrt{1 - u^2/c^2}$.
- If we change to $U' = \Lambda^\mu{}_\nu U^\nu$,

$$\gamma_{u'} = \gamma\gamma_u(1 - \frac{uv}{c^2}\cos(\theta)), \quad \text{with} \quad \theta = \angle(\vec{u}, \vec{v}). \tag{6}$$

- In the system that is bound to a particle with velocity $\vec{u}$, $U' = c(1, \vec{0})$, for a 4V $A^\mu$,

$$A'^{\,0} = -\frac{1}{c} U^\mu A_\mu = -\frac{1}{c} U'^\mu A'_\mu .. \tag{7}$$



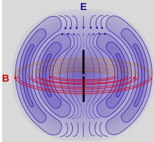
# 1.1- Quadrivectors

- We note that the phase of a plane wave must be Lorentz invariant because the simultaneous cancellation of $\vec{E}$ and $\vec{B}$ in one system implies their cancellation in any other system.
- Let's introduce $k^\mu = (\omega/c, \vec{k})$:

$$k^\mu x_\mu = \vec{k} \cdot \vec{x} - \omega t = \text{ invariant } \Rightarrow \; k^\mu \text{ is 4V.} \tag{8}$$

- We can use Eq. 7 to deduce the relativistic Doppler effect (tarea)

$$c k'^{\,0} = \omega' = -U^\mu k_\mu = \omega\gamma(1 - \frac{v}{c}\cos(\theta)). \tag{9}$$



## 1.1- Quadrivectors

- The gradient operator is another example of 4V. If $\lambda$ is a scalar invariant, then

$$\lambda_{,\mu} \equiv \frac{\partial \lambda}{\partial x^{\mu}} \quad \text{is a covariant 4V, and} \tag{10}$$

$$\lambda^{,\mu} \equiv \frac{\partial \lambda}{\partial x_{\mu}} \quad \text{is a contravariant 4V.} \tag{11}$$

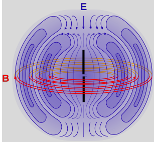
- Proof: from $x^{\nu} = \tilde{\Lambda}_{\mu}{}^{\nu} x'^{\mu}$, we have that $\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \tilde{\Lambda}_{\mu}{}^{\nu}$, and since $\lambda' = \lambda$,

$$\lambda'_{,\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \lambda}{\partial x^{\nu}}. \tag{12}$$

- We extend the properties of 4Vs to tensors in general: a tensor of orden $n$ transforms as the product of $n$ 4Vs.
- For example,

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} T^{\sigma\tau},$$

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\sigma} \tilde{\Lambda}_{\nu}{}^{\tau} T^{\sigma}{}_{\tau}.$$



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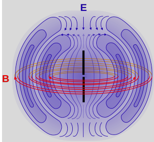


# 1.2- Covariance in electrodynamics

- Charge conservation, $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$, can be written as

$$J^{\mu}{}_{,\mu} = 0, \quad \text{using the four-current} \quad J^{\mu} = (\rho\, c, \vec{J}). \tag{13}$$

- In the Lorentz Gauge , and using CGS units,

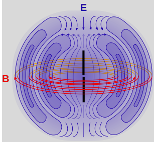
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} = \partial_\alpha\partial^\alpha\vec{A}, \tag{14}$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho = \partial_\alpha\partial^\alpha\Phi. \tag{15}$$

- With $A^{\mu} = (\Phi, \vec{A})$,

$$A^{\beta,\alpha}_{,\alpha} = -\frac{4\pi}{c}J^{\beta}, \quad \text{in which} \quad A^{\beta,\alpha}_{,\alpha} = \frac{\partial^2}{\partial x_\alpha x^\alpha}A^{\beta}. \tag{16}$$

- The Lorentz gauge $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$ can be written simply as $A^{\alpha}{}_{,\alpha} = 0$.



## 1.2- Covariance in electrodynamics

- In order to write the Maxwell equations in their covariant form, we introduce the field tensor

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{17}$$

- With $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ (tarea): 2

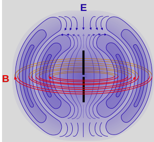
$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{18}$$

- The Maxwell equations $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and $\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$ can be written (tarea)

$$F_{\mu\nu}{}^{,\nu} = \frac{4\pi}{c} J_{\mu}. \tag{19}$$

- The ‘internal’ equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ are written (tarea)

$$F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0. \tag{20}$$



# 1.2- Covariance in electrodynamics

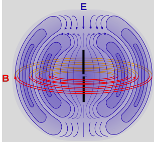
- We use the covariance of $F_{\mu\nu}$ to infer the transformation laws for the fields $\vec{E}$ y $\vec{B}$:

$$F'_{\mu\nu} = \tilde{\Lambda}_{\mu}^{\ \alpha}\tilde{\Lambda}_{\nu}^{\ \beta}F_{\alpha\beta}. \tag{21}$$

- In terms of components we get (tarea):

$$\begin{aligned} E'_{\parallel} &= E_{\parallel}, & B'_{\parallel} &= B_{\parallel}, \\ E'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}), & B'_{\perp} &= \gamma(\vec{B}_{\perp} + \vec{\beta} \times \vec{E}). \end{aligned} \tag{22}$$

- We see that $\vec{E}$ and $\vec{B}$ get mixed up, and if $\vec{B} = 0$ in $\mathcal{S}$, then when changing to $\mathcal{S}'$ we have $\vec{B}' \neq 0$.



## 1.2- Covariance in electrodynamics

- In order to extend the Lorentz force, we introduce the momentum quadrivector (four-momentum) $P^\mu = m_\circ U^\beta$, where $m_\circ$ is the rest mass. We write $P^\mu = (E/c, \vec{P})$, in which $E$ is the total energy of the particle (which is $E = m_\circ c^2$ at rest).
- The acceleration 4V (four-acceleration) is

$$a^\mu = \frac{dU^\mu}{d\tau}, \tag{23}$$

and in order to recover Newton's 2nd law in the non-relativistic limit, the four-force must be

$$F^\mu = m_\circ a^\mu = \frac{dP^\mu}{d\tau}. \tag{24}$$

- We write the 4-Lorentz force with

$$F^\mu = \frac{q}{c} F^\mu{}_\nu U^\nu. \tag{25}$$

- in components, (tarea) $\vec{F} = q(\frac{\vec{v}}{c} \times \vec{B}) + q\vec{E}$.