Part II Electromagnetic wave propagation

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1 Electromagnetic waves

1.1 Maxwell Equations

• In the MKS system (or S.I.), the equations of electrodynamics are, :

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},\tag{3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$
 (4)

- For linear media, $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.
- In vacuum, $\epsilon = \epsilon_{\circ}$ and $\mu = \mu_{\circ}$.

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1.2 Electrodynamic potentials

• Since $\vec{\nabla} \cdot \vec{B} = 0$, we have

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$
 (5)

• For \vec{E} , we use Eq. 5 and Eq. 3:

$$\vec{\nabla} \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)}_{\equiv -\vec{\nabla}\Phi} = 0, \Rightarrow$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$
(6)

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1.3 Wave equations

- We want to write equations that determine the electrodynamic potentials \vec{A} and Φ .
- Using the Maxwell Equations in vacuum to connect directly with \vec{E} and \vec{B} , we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_{\circ}} \Rightarrow \nabla^2 \Phi + \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{A} \right) = \frac{\rho}{\epsilon_{\circ}}, \tag{7}$$

$$\vec{\nabla} \times \frac{1}{\mu_{\circ}} \vec{B} = \vec{J} + \frac{1}{\epsilon_{\circ}} \frac{\partial \vec{E}}{\partial t} \Rightarrow$$

$$\nabla^{2} \vec{A} - \frac{1}{c^{2}} \frac{\partial^{2} \vec{A}}{\partial t^{2}} - \vec{\nabla} \underbrace{\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^{2}} \frac{\partial \Phi}{\partial t}\right)}_{\text{term for the Lorentz condition}} = -\mu_{\circ} \vec{J}. \quad (8)$$

• If the term highlighted in Eq. 8 is null, which is called the Lorentz Condition,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \tag{9}$$

then we recover the wave equation for the potentials:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_{\circ}}, \tag{10}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_\circ \vec{J}.$$
 (11)

• To fulfill the Lorentz condition, we use the freedom of gauge:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda,$$
 (12)

which leaves invariant $\vec{B} = \vec{\nabla} \times \vec{A}$.

• To also preserve $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t$, it is necessary that

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}.$$
 (13)

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- If \vec{A} and Φ both fulfill the general potential equations (Eqs. 8 and 7), but do not fulfill the Lorentz condition, then we can search for $\Lambda(\vec{x}, t)$ so that $\vec{A'}$ and Φ' do satisfy the Lorentz condition.
- Injecting Eqs. 12 and 13 in Eq. 9, we reach an equation for $\Lambda(\vec{x}, t)$:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t} = 0, \qquad (14)$$

which is essentially a wave equation with a source term, i.e. exactly the type of equations that we will propose solutions for.

• Independently of the Lorentz condition, we can manipulate the Maxwell equations to reach (tarea):

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_o} \left(-\vec{\nabla} \rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \tag{15}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = -\mu_\circ \vec{\nabla} \times \vec{J}, \qquad (16)$$

which are both wave equations with source terms.

• Away from the sources, i.e. in vacuum, both equations become the homogeneous wave equation.

1.4 Poynting's theorem

- The power exerted by the electromagnetic force $\vec{\mathcal{F}} = q\vec{v} \times \vec{B} + q\vec{E}$ on a single charge q with velocity \vec{v} is $\vec{v} \cdot \vec{\mathcal{F}} = q\vec{v} \cdot \vec{E}$.
- The power exerted on the charge density distribution ρ and on the current density distribution $\vec{J} = \rho \vec{v}$ inside a volume $d\mathcal{V}$ is thus

$$dP = \vec{J} \cdot \vec{E} \, d\mathcal{V}$$

• The total power exerted by the (\vec{E}, \vec{B}) field on the charges inside a volume \mathcal{V} is

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3 x. \tag{17}$$

• We want to connect P with the energy stored in the fields. Using the Ampère-Maxwell equation (Eq. 4) we solve for \vec{J} , and following standard handling (tarea),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x.$$
(18)

• Now with the induction law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Eq. 3),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x.$$
(19)

• Remembering that for a linear medium $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})$, and $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$, we reach

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3 x = -\int_{\mathcal{V}} \left[\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right] d^3 x, \tag{20}$$

where we recognize

$$u = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H},\tag{21}$$

and

$$\vec{S} = \vec{E} \times \vec{H}.$$
 (22)

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• For any volume \mathcal{V} , we conclude that

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}.$$
(23)

• In the same way as for energy conservation, Eq. 23, we can also write the equation for the conservation of linear momentum. Newton's 2nd law for the variation of linear momentum $\delta \vec{p}_{mec}$ inside a volume $\delta \mathcal{V}$ is:

$$\frac{d\,\delta\vec{p}_{\rm mec}}{dt} = \rho\vec{E}\delta\mathcal{V} + \rho\vec{v}\times\vec{B}\delta\mathcal{V}.$$
(24)

• In total,

$$\frac{d\vec{p}_{\rm mec}}{dt} = \int_{\mathcal{V}} d^3x (\rho \vec{E} + \rho \vec{v} \times \vec{B}).$$
(25)

• Using Maxwell's equation to replace ρ and \vec{J} , we reach (tarea):

$$\frac{d}{dt}(\vec{p}_{\rm mec} + \vec{p}_{\rm fields})|_i = \sum_j \int_{\mathcal{V}} d^3x \frac{\partial T_{ij}}{\partial x_j},\tag{26}$$

with the following notations:

$$\vec{p}_{\text{fields}} = \int \epsilon_{\circ} (\vec{E} \times \vec{B}) d^3 x = \frac{1}{c^2} \int d^3 x \vec{S}, \qquad (27)$$

which we associate to the momentum in the fields since it fulfills a similar role as $\vec{p}_{\rm mec}$, and

$$T_{ij} = \epsilon_{\circ} \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \left(\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{ij} \right], \qquad (28)$$

which is the tensor of electromagnetic tensions.

• For each component *i* the integrand of Eq. 26 (involving T_{ij}) can be seen as a divergence, so

$$\frac{d}{dt}(\vec{p}_{\rm mec} + \vec{p}_{\rm fields})|_i = \oint_{\mathcal{S}} \sum_j T_{ij} n_j d\mathcal{A},$$
(29)

where we recognize a flux integral over the surface bounding the volume \mathcal{V} .

2 Wave propagation in vacuum

2.1 Spectral decomposition

• In the absence of sources, if we decompose

$$\vec{E}(\vec{x},t) = \frac{1}{2\pi} \int d\omega \vec{E}(\vec{x},\omega) e^{i\omega t},$$
(30)

the Maxwell equations yield

$$\left(\nabla^2 + \mu\epsilon\omega^2\right) \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} = 0.$$
(31)

- If ϵ and μ are both real, the solutions are $e^{\pm ikx}$, with $k = \sqrt{\mu\epsilon}\omega$
- We define the phase velocity $v_{\phi} = \frac{\omega}{k} = \frac{c}{n}$, where $n = \sqrt{\frac{\mu\epsilon}{\mu_{\circ}\epsilon_{\circ}}}$ is the refraction index.
- In general,

$$\left\{\begin{array}{c}E_i\\B_i\end{array}\right\} = \frac{1}{2\pi} \int d\omega \left\{\begin{array}{c}\mathcal{E}_i\\\mathcal{B}_i\end{array}\right\} e^{\pm i\vec{k}\cdot\vec{x}-i\omega t},\tag{32}$$

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• We recognize d'Alembert's solution for the wave equation,

$$\left\{ \begin{array}{c} E_i \\ B_i \end{array} \right\} = \frac{1}{2\pi} \int d\omega \left\{ \begin{array}{c} \mathcal{E}_i \\ \mathcal{B}_i \end{array} \right\} e^{\pm ik(\hat{n}\cdot\vec{x} - v_{\phi}t)},$$
(33)

where each component *i* has a form $f(\hat{n} \cdot \vec{x} - v_{\phi}t) + g(\hat{n} \cdot \vec{x} + v_{\phi}t)$, and where \hat{n} is the direction of propagation.

- Using Maxwell's equations (tarea), $\hat{n} \cdot \vec{\mathcal{E}} = 0$, $\hat{n} \cdot \vec{\mathcal{B}} = 0$ and $\vec{\mathcal{B}} = \frac{n}{c}\hat{n} \times \vec{\mathcal{E}}$.
- For harmonic fields it is customary to use complex notation (because of the spectral decomposition), so that $\vec{S} = \Re(\vec{E}) \times \Re(\vec{H})$.
- In general for products of the type

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^*b + abe^{-2i\omega t}),\tag{34}$$

it is also customary to take time averages $\langle (\cdots) \rangle_T = \lim_{T \to \infty} \frac{1}{T} \int_0^\infty (\cdots) dt$, and

$$\langle \Re(ae^{-i\omega t})\Re(be^{-i\omega t})\rangle = \frac{1}{2}\Re(a^*b).$$
(35)

• We therefore have

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathcal{E}|^2 \hat{n}.$$
(36)

And similarly,

$$\langle u \rangle = \frac{1}{4} (\epsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*) = \frac{\epsilon}{2} |\mathcal{E}|^2.$$
(37)

• Finally, $\langle \vec{S} \rangle = v_{\phi} u \hat{n}$.

2.2 Connection with radiative transfer

- We can now see that the concept of rays associated to the radiative transfer equation, which describes the transport of radiation in a straight line, is connected to the idea of a plane monochromatic wave with direction of propagation \vec{k} .
- For a plane wave then $\langle \vec{S} \rangle = v_{\phi} u \hat{k}$ is the flux of energy in direction \hat{k} .
- In radiative transfer notation, the flux density in direction k_{\circ} would be

$$F_{\nu}(\vec{x}) = \int d\Omega I_{\nu}(\hat{k}, \vec{x}) \, \hat{k} \cdot \hat{k}_{\circ}.$$
(38)

• Therefore the specific intensity field for a monochromatic plane wave is

$$I_{\nu}(\hat{k}) = \|\vec{S}\| \,\delta(\hat{k} - \hat{k}_{\circ}). \tag{39}$$

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2.3 Polarization

• In summary, the electric field of a monochromatic wave can be decomposed in two linearly polarized waves,

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2) e^{i(\vec{k} \cdot \vec{x} - \omega t)},$$
(40)

whose total describes, in general, an eliptically polarized wave.

• With a change of vectorial basis to $\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$, we can also decompose \vec{E} in two circularly polarized waves,

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_{+}E_{+} + \hat{\epsilon}_{-}E_{-})e^{i(\vec{k}\cdot\vec{x}-\omega t)}.$$
(41)

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• With the notation

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \quad E_2 = \mathcal{E}_2 e^{i\phi_2}, \\ E_+ = \mathcal{E}_+ e^{i\phi_+}, \quad E_- = \mathcal{E}_- e^{i\phi_-},$$

we have

$$\begin{cases} \text{linear polarization} : \phi_2 - \phi_1 = 0. \\ \text{circular polarization} : |\phi_2 - \phi_1| = \frac{\pi}{2} \text{ and } \mathcal{E}_2 = \mathcal{E}_1. \\ \text{the general case is eliptical, with} : \tan(\chi) = \frac{\mathcal{E}_1}{\mathcal{E}_2} \frac{\cos(\phi_1)}{\cos(\phi_2)}. \end{cases}$$



• It is customary to use the Stokes parameters to characterize the polarization state of **monochromatic light**:

$$I = E_{1}E_{1}^{*} + E_{2}E_{2}^{*} = \mathcal{E}_{1}^{2} + \mathcal{E}_{2}^{2},$$

$$Q = E_{1}E_{1}^{*} - E_{2}E_{2}^{*} = \mathcal{E}_{1}^{2} - \mathcal{E}_{2}^{2},$$

$$U = E_{1}E_{2}^{*} - E_{2}E_{1}^{*} = 2\mathcal{E}_{1}\mathcal{E}_{2}\cos(\phi_{2} - \phi_{1}),$$

$$V = i(E_{1}E_{2}^{*} - E_{2}E_{1}^{*}) = 2\mathcal{E}_{1}\mathcal{E}_{2}\sin(\phi_{2} - \phi_{1}).$$
(42)

- We see that Stokes I (the total "radiance") is $I \propto |\vec{S}|$, Q and U measure linear polarization, while V measure circular polarization. In order to make this obvious it is best to use mental experiments with polarizors that select specific types of polarization (see class).
- For a strictly monochromatic wave, it follows that

$$I^2 = Q^2 + U^2 + V^2. (43)$$

2.4 Quasi-monochromatic waves

• In order to obtain $\vec{E}(\vec{x}, \omega)$, we need to know $\vec{E}(t)$ for all t, since

$$\vec{E}(\vec{x},\omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{x},t)e^{i\omega t}dt.$$
(44)

• So in practice, we treat E_1 and E_2 as random variables, i.e. for a wave in vacuum, described by Eq. 40,

$$\vec{E}(\vec{x},t) = (E_1(t)\hat{e}_1 + E_2(t)\hat{e}_2)e^{i(\vec{k}\cdot\vec{x}-\omega t)}.$$
(45)

Alternatively we can also replace the time dependence in Eq. 45 with a probability density, which itself may depend on time.

- To fix ideas, let's remember that $\Delta t \Delta \omega = 1$ for Gaussian spectra, where Δt is the 'coherence time', and $\Delta \omega$ is the 'bandwidth' of the quasi-monochromatic wave.
- In order to measure the Stokes parameters, we need averages of the kind

$$\langle E_1 E_2^* \rangle = \lim_{T \to \infty} \frac{1}{T} \int dt E_1(t) E_2^*(t) dt.$$
(46)

• We therefore have

$$\begin{split} \langle Q^2 \rangle + \langle U^2 \rangle + \langle V^2 \rangle &= \langle I^2 \rangle - \\ & 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1 \mathcal{E}_2 e^{i(\phi_2 - \phi_1)} \rangle \langle \mathcal{E}_1 \mathcal{E}_2 e^{-i(\phi_2 - \phi_1)} \rangle \\ &= \langle I^2 \rangle - \\ & 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle), \end{split}$$
(47)

and, by Shwartz' inequality $(\langle ab \rangle \leq \langle a \rangle \langle b \rangle)$,

$$I^2 \ge Q^2 + U^2 + V^2. \tag{48}$$

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- For a wave with a single and constant eliptical polarization state, then the equality holds in Eq. 48.
- On the other hand, for a completely unpolarized wave, Q = U = V = 0.
- The Stokes parameters are additive. Proof: consider a sum of N different waves

$$\vec{E} = \sum_{k=1}^{N} \vec{E}^{k} = \sum (\hat{\epsilon}_{1} E_{1}^{k} + \hat{\epsilon}_{2} E_{2}^{k}) e^{i(\vec{k} \cdot \vec{x} - \omega t)}.$$
(49)

Because each $E_i^k(t)$ is statistically independent, $\langle E_i^k E_j^{l*} \rangle = \delta_{kl} \langle E_i^k E_j^{k*} \rangle$, and

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \sum_{k} \begin{pmatrix} I_{k} \\ Q_{k} \\ U_{k} \\ V_{k} \end{pmatrix}.$$
(50)

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• We can therefore decompose an arbitrary set of Stokes parameters in

$$\begin{pmatrix} I\\Q\\U\\V\\V \end{pmatrix} = \overbrace{\begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2}\\0\\0 \end{pmatrix}}^{\text{unpol}} + \overbrace{\begin{pmatrix} 0\\0\\0 \end{pmatrix}}^{\text{pol}} + \overbrace{\begin{pmatrix} \sqrt{Q^2 + U^2 + V^2}\\0\\V \end{pmatrix}}^{\text{pol}} \cdot (51)$$

- The first term 'unpol' is completely unpolarized since Q = U = V = 0, while the second term 'pol' is completely polarized since it satisfies $I^2 = Q^2 + U^2 + V^2$ (Eq. 43).
- The total polarized intensity of a wave train is thus be $I^{\text{pol}} = \sqrt{Q^2 + U^2 + V^2}$.
- We define the polarization fraction as

$$\Pi = \frac{I^{\text{pol}}}{I}.$$
(52)

3 Wave propagation in a medium

3.1 Constitutive equations

• Each monochromatic component of the field \vec{E}, \vec{B} must fulfill the following constitutive relations:

$$\vec{P} = \epsilon_{\circ} \chi \vec{E} \longrightarrow \vec{P}(\omega) = \epsilon_{\circ} \chi(\omega) \vec{E}(\omega),
\vec{B} = \mu \vec{H} \longrightarrow \vec{B}(\omega) = \mu(\omega) \vec{H}(\omega),
\vec{J} = \sigma \vec{E} \longrightarrow \vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega),$$
(53)

in which we have added Ohm's law.

- We note that $\chi(-\omega) = \chi^*(\omega)$, so that $\chi(t) = \frac{1}{2\pi} \int d\omega \chi(\omega) \exp(-i\omega t)$ be real (and similarly for μ and σ).
- The Fourier convolution theorem states that if $X(\omega) = Y(\omega)Z(\omega)$, then

$$X(t) = \int_{-\infty}^{\infty} Y(t - t')Z(t')dt',$$
(54)

where $Y(t) = \frac{1}{2\pi} \int d\omega Y(\omega) \exp(-i\omega t)$, etc..

• Applying the convolution theorem to χ (for example),

$$P(t) = \int_{-\infty}^{\infty} G(t - t') E(t') dt', \text{ with}$$
$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_{\circ} \chi(\omega) e^{-i\omega t} d\omega.$$
(55)

- We see that P(t) depends on the history of $\vec{E}(t')$, which bears physical sense only in the past, for t' < t, so G(t) = 0 if t < 0. We will use this property in the next section.
- This time we write the monochromatic wave as

$$\vec{E}(t) = \vec{A}\cos(\omega_{\circ}t) + \vec{B}\sin(\omega_{\circ}t) = \Re(\vec{E}_{c}(t)),$$
(56)

with $\vec{E}_c = (\vec{A} - i\vec{B})(\cos(\omega_\circ t) + i\sin(\omega_\circ t)).$ • In the Fourier plane,

$$E(\omega) = \pi \left[(A + iB)\delta(\omega - \omega_{\circ}) + (A - iB)\delta(\omega + \omega_{\circ}) \right].$$
(57)

• We can evaluate

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_{\circ} \chi(\omega) E(\omega) e^{-i\omega t} d\omega$$
$$= \Re \left[\frac{\epsilon_{\circ}}{2} (A - iB) \chi(\omega_{\circ}) e^{-i\omega_{\circ} t} \right] = \Re [P_{c}(t)], \quad (58)$$

using $\chi(-\omega) = \chi^*(\omega)$, and where $P_c = \epsilon_{\circ}\chi(\omega_{\circ})E_c(t)$.

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• With the spectral decomposition of the constitutive relations we can rewrite the Maxwell equations in their harmonic versions. In the absence of free charges,

$$\vec{\nabla} \cdot \vec{E}(\omega) = 0, \quad \vec{\nabla} \times \vec{E}(\omega) = -i\omega\mu(\omega)\vec{H}(\omega), \vec{\nabla} \cdot \vec{H}(\omega) = 0, \quad \vec{\nabla} \times \vec{H}(\omega) = -i\omega\epsilon(\omega)\vec{E}(\omega),$$
(59)

where (tarea)

$$\epsilon(\omega) = \epsilon_{\circ}(1 + \chi(\omega)) + i \frac{\sigma(\omega)}{\omega}.$$
(60)

• Note that both susceptibility and conductivity contribute to the imaginary part of ϵ :

$$\Im(\epsilon) = \epsilon_{\circ}\Im(\chi) + \Re(\sigma/\omega).$$
(61)

3.2 Kramers-Kronig relations

• From physical considerations we can anticipate that the induced P(t) depends on the history of the applied field, or

$$\vec{P}(t) = \int_{-\infty}^{\infty} G(t, t') \vec{E}(t') dt'$$
(62)

(note difference with Eq. 55).

- Let's assume that $\vec{E} = \delta(t t_{\circ})\vec{E}_{\circ}$. Then $\vec{P}(t) = G(t, t_{\circ})\vec{E}_{\circ}$, and G is the polarization resulting from a delta-unitary electric field.
- If the properties of the medium do not change in time, $G(t, t_{\circ}) = G(t t_{\circ})$, and we recover Eq. 55.
- Causality requires that $G(\tau) = 0$ if $\tau < 0$, so

$$\epsilon_{\circ}\chi(\omega) = \int_{0}^{\infty} dt G(t) e^{i\omega t}.$$
(63)

• We extend Eq. 63 to the complex plane with $\tilde{\omega} = \omega_R + i\omega_I$, where $\omega_I > 0$.

$$\epsilon_{\circ}\chi(\tilde{\omega}) = \int_{0}^{\infty} dt G(t) e^{i\tilde{\omega}t}.$$
(64)

- If $\int_0^\infty |G(t)| dt$ converges, so does $\int_0^\infty G(t) e^{i\tilde{\omega}t} dt$, and $\chi(\tilde{\omega})$ is analytical in the superior \mathbb{C} plane ($\omega_I > 0$).
- Therefore χ(ω̃)/(ω̃ − ω) is analytical except in the pole ω̃ = ω, where ω is a point along the real axis.

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• We can apply the Kramers-Kronig theorem (proof: see Bohren & Huffman, Sec. 2.3.2), which gives

$$i\pi\chi(\omega) = P \int_{-\infty}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega,$$
(65)

where P indicates Cauchy's 'principal value'

$$P \int_{-\infty}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega = \lim_{a \to 0} \left(\int_{-\infty}^{\omega - a} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega + \int_{\omega + a}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega \right).$$
(66)

• Using that $\chi^*(\Omega) = \chi(-\Omega)$ we can restrict the integration to $\Omega > 0$, and use $\chi = \chi_R + i\chi_I$ to rewrite Eq. 66:

$$\chi_R(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\Omega \,\chi_I(\Omega)}{\Omega^2 - \omega^2} d\Omega, \tag{67}$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\chi_R(\Omega)}{\Omega^2 - \omega^2} d\Omega.$$
 (68)

• Similar relationships exists for μ y σ .

3.3 Monochromatic waves

• We now extend the monochromatic waves to homogeneous media. We inject

$$\vec{E}_c = \vec{E}_{\circ} e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \text{ and } \vec{H}_c = \vec{H}_{\circ} e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{69}$$

- into Maxwell's equations. Allowing for $\vec{k} \in \mathbb{C}$, $\vec{k} = \underbrace{(k_R + ik_I)}_{k} \hat{e}$, $\vec{E}_c = \vec{E}_o e^{-\vec{k_I} \cdot \vec{x}} e^{i(\vec{k_R} \cdot \vec{x} - \omega t)}.$ (70)
- The harmonic Maxwell equations (Eqs. 59) yield:

$$\vec{k} \cdot \vec{E}_{\circ} = 0 \qquad \vec{k} \cdot \vec{H}_{\circ}(\omega) = 0$$

$$\vec{k} \times \vec{E}_{\circ} = \omega \mu \vec{H}_{\circ}, \quad \vec{k} \times \vec{H}_{\circ} = -\omega \epsilon \vec{E}_{\circ}.$$
(71)

• And with $\vec{k} \cdot \vec{k} = \omega^2 \epsilon \mu$,

$$k_R^2 - k_I^2 + 2i\vec{k}_I \cdot \vec{k}_R = \omega^2 \epsilon \mu$$
 (tarea). (72)

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• For a homogeneous wave (no free charges),

$$\vec{k} = \underbrace{(k_R + ik_I)}_k \hat{e},$$

and $k = \omega N/c$, where N is the complex refractive index,

$$N = c\sqrt{\epsilon\mu} = \sqrt{\frac{\epsilon\mu}{\epsilon_{\circ}\mu_{\circ}}}.$$

- We set $N = n + i\kappa$, where n and κ are both $\in \mathbb{R}^+$.
- Eq. 70 gives:

$$\vec{E}_c = \vec{E}_o e^{-\frac{2\pi}{\lambda}\kappa z} e^{i(\frac{2\pi n z}{\lambda} - i\omega t)}.$$
(73)

- \Rightarrow the imaginary part of N corresponds to absorption.
- We can apply the Kramers-Kronig relations to (N(ω)−1) (the −1 is motivated by lim_{ω→∞} N(ω) = 1):

$$n(\omega) - 1 = \frac{2}{\pi} P \int_0^\infty \frac{\Omega \kappa(\Omega)}{\Omega^2 - \omega^2} d\Omega$$

$$\kappa(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{n(\Omega)}{\Omega^2 - \omega^2} d\Omega$$
(74)

• We see that the absorption in a medium is also related to the real refractive index.

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