Radiative Processes

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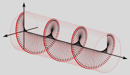
Part II

Electromagnetic wave propagation



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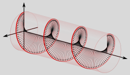
1. **Electromagnetic waves**
   - Maxwell Equations
   - Electrodynamic potentials
   - Wave equations
   - Poynting's theorem

2. **Wave propagation in vacuum**
   - Spectral decomposition
   - Connection with radiative transfer
   - Polarization
   - Quasi-monochromatic waves

3. **Wave propagation in a medium**
   - Constitutive equations
   - Kramers-Kronig relations
   - Monochromatic waves



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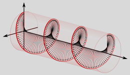


1.1-Maxwell Equations

- In the MKS system (or S.I.), the equations of electrodynamics are, :

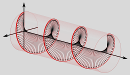
$$\vec{\nabla} \cdot \vec{D} = \rho, \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (3)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (4)$$

- For linear media, $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.
- In vacuum, $\epsilon = \epsilon_\circ$ and $\mu = \mu_\circ$.



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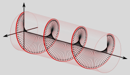

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1.2-Electrodynamic potentials

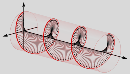
- Since $\vec{\nabla} \cdot \vec{B} = 0$, we have

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{5}$$

- For $\vec{E}$, we use Eq. 5 and Eq. 3:

$$\vec{\nabla} \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right)}_{\equiv -\vec{\nabla}\Phi} = 0, \;\Rightarrow$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{6}$$



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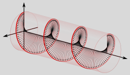


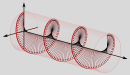
1.3-Wave equations

- We want to write equations that determine the electrodynamic potentials $\vec{A}$ and Φ.
- Using the Maxwell Equations in vacuum to connect directly with $\vec{E}$ and $\vec{B}$, we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_\circ} \;\Rightarrow\; \nabla^2 \Phi + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\rho}{\epsilon_\circ}, \tag{7}$$

$$\vec{\nabla} \times \frac{1}{\mu_\circ}\vec{B} = \vec{J} + \frac{1}{\epsilon_\circ}\frac{\partial \vec{E}}{\partial t} \;\Rightarrow$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \underbrace{\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)}_{\text{term for the Lorentz condition}} = -\mu_\circ \vec{J}. \tag{8}$$



1.3-Wave equations

- If the term highlighted in Eq. 8 is null, which is called the *Lorentz Condition*,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0, \qquad (9)$$

  then we recover the wave equation for the potentials:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_\circ}, \qquad (10)$$

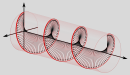
$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_\circ \vec{J}. \qquad (11)$$

- To fulfill the Lorentz condition, we use the freedom of gauge:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \qquad (12)$$

  which leaves invariant $\vec{B} = \vec{\nabla} \times \vec{A}$.
- To also preserve $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t$, it is necessary that

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \qquad (13)$$

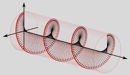


1.3-Wave equations

- If $\vec{A}$ and Φ both fulfill the general potential equations (Eqs. 8 and 7), but do not fulfill the Lorentz condition, then we can search for $\Lambda(\vec{x}, t)$ so that $\vec{A}'$ and Φ' do satisfy the Lorentz condition.
- Injecting Eqs. 12 and 13 in Eq. 9, we reach an equation for $\Lambda(\vec{x}, t)$:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t} = 0, \tag{14}$$

which is essentially a wave equation with a source term, i.e. exactly the type of equations that we will propose solutions for.



1.3-Wave equations

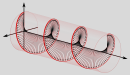
- Independently of the Lorentz condition, we can manipulate the Maxwell equations to reach (tarea):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_\circ}\left(-\vec{\nabla}\rho - \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t}\right), \qquad (15)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_\circ \vec{\nabla} \times \vec{J}, \qquad (16)$$

which are both wave equations with source terms.

- Away from the sources, i.e. in vacuum, both equations become the homogeneous wave equation.



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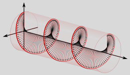

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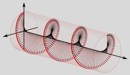
1.4 Poynting's theorem

- The power exerted by the electromagnetic force $\vec{\mathcal{F}} = q\vec{v} \times \vec{B} + q\vec{E}$ on a single charge q with velocity $\vec{v}$ is $\vec{v} \cdot \vec{\mathcal{F}} = q\vec{v} \cdot \vec{E}$.
- The power exerted on the charge density distribution ρ and on the current density distribution $\vec{J} = \rho\vec{v}$ inside a volume $d\mathcal{V}$ is thus
$$dP = \vec{J} \cdot \vec{E}\, d\mathcal{V}$$
- The total power exerted by the $(\vec{E}, \vec{B})$ field on the charges inside a volume $\mathcal{V}$ is
$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x. \tag{17}$$



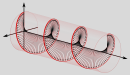
1.4- Poynting's theorem

- We want to connect P with the energy stored in the fields. Using the Ampère-Maxwell equation (Eq. 4) we solve for $\vec{J}$, and following standard handling (tarea),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \tag{18}$$

- Now with the induction law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Eq. 3),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \tag{19}$$



1.4 Poynting's theorem

- Remembering that for a linear medium $\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B})$, and $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$, we reach

$$P=\int_{\mathcal{V}}\vec{J}\cdot\vec{E}d^3x=-\int_{\mathcal{V}}\left[\frac{\partial u}{\partial t}+\vec{\nabla}\cdot\vec{S}\right]d^3x, \tag{20}$$

where we recognize

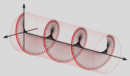
$$u=\frac{1}{2}\vec{E}\cdot\vec{D}+\frac{1}{2}\vec{B}\cdot\vec{H}, \tag{21}$$

and

$$\vec{S}=\vec{E}\times\vec{H}. \tag{22}$$

- For any volume $\mathcal{V}$, we conclude that

$$\frac{\partial u}{\partial t}+\vec{\nabla}\cdot\vec{S}=-\vec{J}\cdot\vec{E}. \tag{23}$$



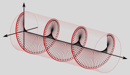
1.4 Poynting's theorem

- In the same way as for energy conservation, Eq. 23, we can also write the equation for the conservation of linear momentum. Newton's 2nd law for the variation of linear momentum $\delta\vec{p}_{\rm mec}$ inside a volume $\delta\mathcal{V}$ is:

$$\frac{d\,\delta\vec{p}_{\rm mec}}{dt} = \rho\vec{E}\delta\mathcal{V} + \rho\vec{v}\times\vec{B}\delta\mathcal{V}. \tag{24}$$

- In total,

$$\frac{d\vec{p}_{\rm mec}}{dt} = \int_{\mathcal{V}} d^3x(\rho\vec{E} + \rho\vec{v}\times\vec{B}). \tag{25}$$



1.4 Poynting's theorem

- Using Maxwell's equation to replace ρ and $\vec{J}$, we reach (tarea):

$$\frac{d}{dt}(\vec{p}_{\mathrm{mec}} + \vec{p}_{\mathrm{fields}})|_i = \sum_j \int_{\mathcal{V}} d^3x \frac{\partial T_{ij}}{\partial x_j}, \tag{26}$$

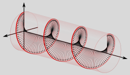
with the following notations:

$$\vec{p}_{\mathrm{fields}} = \int \epsilon_\circ (\vec{E} \times \vec{B}) d^3x = \frac{1}{c^2} \int d^3x \vec{S}, \tag{27}$$

which we associate to the momentum in the fields since it fulfills a similar role as $\vec{p}_{\mathrm{mec}}$, and

$$T_{ij} = \epsilon_\circ \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \left(\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B} \right) \delta_{ij} \right], \tag{28}$$

which is the tensor of electromagnetic tensions.

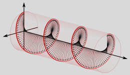


1.4- Poynting's theorem

- For each component i the integrand of Eq. 26 (involving T_{ij}) can be seen as a divergence, so

$$\frac{d}{dt}(\vec{p}_{\rm mec} + \vec{p}_{\rm fields})|_i = \oint_{\mathcal{S}} \sum_j T_{ij} n_j d\mathcal{A}, \tag{29}$$

where we recognize a flux integral over the surface bounding the volume $\mathcal{V}$.



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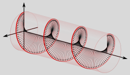

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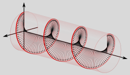


2.1 Spectral decomposition

- In the absence of sources, if we decompose

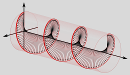
$$\vec{E}(\vec{x},t) = \frac{1}{2\pi}\int d\omega \vec{E}(\vec{x},\omega)e^{iwt}, \tag{30}$$

the Maxwell equations yield

$$(\nabla^2 + \mu\epsilon\omega^2)\left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} = 0. \tag{31}$$

- If ϵ and μ are both real, the solutions are $e^{\pm ikx}$, with $k = \sqrt{\mu\epsilon}\omega$
- We define the phase velocity $v_\phi = \frac{\omega}{k} = \frac{c}{n}$, where $n = \sqrt{\frac{\mu\epsilon}{\mu_\circ\epsilon_\circ}}$ is the refraction index.
- In general,

$$\left\{ \begin{array}{c} E_i \\ B_i \end{array} \right\} = \frac{1}{2\pi}\int d\omega \left\{ \begin{array}{c} \mathcal{E}_i \\ \mathcal{B}_i \end{array} \right\} e^{\pm i\vec{k}\cdot\vec{x} - iwt}, \tag{32}$$



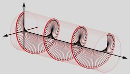
2.1 Spectral decomposition

- We recognize d'Alembert's solution for the wave equation,

$$\left\{ \begin{array}{c} E_i \\ B_i \end{array} \right\} = \frac{1}{2\pi} \int d\omega \left\{ \begin{array}{c} \mathcal{E}_i \\ \mathcal{B}_i \end{array} \right\} e^{\pm ik(\hat{n}\cdot\vec{x} - v_\phi t)}, \qquad (33)$$

  where each component i has a form $f(\hat{n}\cdot\vec{x} - v_\phi t) + g(\hat{n}\cdot\vec{x} + v_\phi t)$, and where $\hat{n}$ is the direction of propagation.

- Using Maxwell's equations (tarea), $\hat{n}\cdot\vec{\mathcal{E}} = 0$, $\hat{n}\cdot\vec{\mathcal{B}} = 0$ and $\vec{\mathcal{B}} = \frac{n}{c}\hat{n}\times\vec{\mathcal{E}}$.



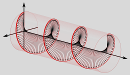
2.1 Spectral decomposition

- For harmonic fields it is customary to use complex notation (because of the spectral decomposition), so that $\vec{S} = \Re(\vec{E}) \times \Re(\vec{H})$.
- In general for products of the type

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^* b + abe^{-2i\omega t}), \tag{34}$$

it is also customary to take time averages $\langle(\cdots)\rangle_T = \lim_{T\to\infty} \frac{1}{T}\int_0^\infty (\cdots)dt$, and (tarea)

$$\langle\Re(ae^{-i\omega t})\Re(be^{-i\omega t})\rangle = \frac{1}{2}\Re(a^* b). \tag{35}$$



2.1- Spectral decomposition

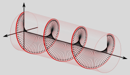
- We therefore have

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathcal{E}|^2 \hat{n}. \tag{36}$$

And similarly,

$$\langle u \rangle = \frac{1}{4} (\epsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*) = \frac{\epsilon}{2} |\mathcal{E}|^2. \tag{37}$$

- Finally, $\langle \vec{S} \rangle = v_\phi u \hat{n}$.



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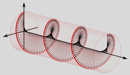


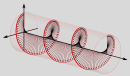
2.2- Connection with radiative transfer

- We can now see that the concept of rays associated to the radiative transfer equation, which describes the transport of radiation in a straight line, is connected to the idea of a plane monochromatic wave with direction of propagation $\vec{k}$.
- For a plane wave $\langle \vec{S} \rangle = v_\phi u \hat{k}$ is the flux of energy in direction $\hat{k}$.
- In radiative transfer notation, the flux density in direction $k_\circ$ would be

$$F_\nu(\vec{x}) = \int d\Omega I_\nu(\hat{k}, \vec{x})\, \hat{k} \cdot \hat{k}_\circ. \tag{38}$$

- Therefore the specific intensity field for a monochromatic plane wave is

$$I_\nu(\hat{k}) = \|\vec{S_\nu}\|\, \delta(\hat{k} - \hat{k}_\circ). \tag{39}$$



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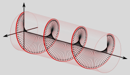


2.3 Polarization

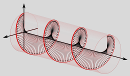
- In summary, the electric field of a monochromatic wave can be decomposed in two linearly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2) e^{i(\vec{k}\cdot\vec{x} - \omega t)}, \tag{40}$$

whose total describes, in general, an eliptically polarized wave.

- With a change of vectorial basis to $\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$, we can also decompose $\vec{E}$ in two circularly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{\epsilon}_+ E_+ + \hat{\epsilon}_- E_-) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \tag{41}$$



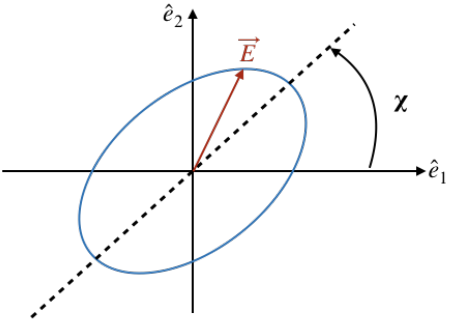
2.3 Polarization

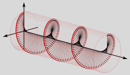
- With the notation

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2},$$
$$E_+ = \mathcal{E}_+ e^{i\phi_+}, \quad E_- = \mathcal{E}_- e^{i\phi_-},$$

we have

$$\begin{cases} \text{linear polarization} : \phi_2 - \phi_1 = 0. \\ \text{circular polarization} : |\phi_2 - \phi_1| = \frac{\pi}{2} \text{ and } \mathcal{E}_2 = \mathcal{E}_1. \\ \text{the general case is eliptical, with} : \mathtt{tan}(\chi) = \frac{\mathcal{E}_1}{\mathcal{E}_2} \frac{\cos(\phi_1)}{\cos(\phi_2)}. \end{cases}$$





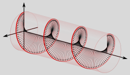
2.3 Polarization

- It is customary to use the Stokes parameters to characterize the polarization state of **monochromatic light**:

$$
\begin{aligned}
I = \quad & E_1 E_1^* + E_2 E_2^* = & \mathcal{E}_1^2 + \mathcal{E}_2^2, \\
Q = \quad & E_1 E_1^* - E_2 E_2^* = & \mathcal{E}_1^2 - \mathcal{E}_2^2, \\
U = \quad & E_1 E_2^* - E_2 E_1^* = & 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_2 - \phi_1), \\
V = \quad & i(E_1 E_2^* - E_2 E_1^*) = & 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_2 - \phi_1).
\end{aligned}
\tag{42}
$$

- We see that Stokes I (the total “radiance”) is $I \propto |\vec{S}|$, Q and U measure linear polarization, while V measure circular polarization. In order to make this obvious it is best to use mental experiments with polarizors that select specific types of polarization (see class).
- For a strictly monochromatic wave, it follows that

$$
I^2 = Q^2 + U^2 + V^2. \tag{43}
$$



Outline

1. Electromagnetic waves
   - Maxwell Equations
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   - Poynting's theorem
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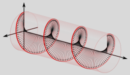


2.4 Quasi-monochromatic waves

- In order to obtain $\vec{E}(\vec{x},\omega)$, we need to know $\vec{E}(\mathrm{t})$ for all t, since

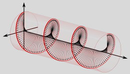
$$\vec{E}(\vec{x},\omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{x},t) e^{i\omega t} dt. \tag{44}$$

- So in practice, we treat E_1 and E_2 as random variables, i.e. for a wave in vacuum, described by Eq. 40,

$$\vec{E}(\vec{x},t) = (E_1(t)\hat{e}_1 + E_2(t)\hat{e}_2) e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{45}$$

  Alternatively we can also replace the time dependence in Eq. 45 with a probability density, which itself may depend on time.

- To fix ideas, let's remember that $\Delta t \Delta\omega = 1$ for Gaussian spectra, where Δt is the 'coherence time', and $\Delta\omega$ is the 'bandwidth' of the quasi-monochromatic wave.



2.4 Quasi-monochromatic waves

- In order to measure the Stokes parameters, we need averages of the kind

$$\langle E_1 E_2^* \rangle = \lim_{T\to\infty} \frac{1}{T} \int dt E_1(t) E_2^*(t) dt. \tag{46}$$

- We therefore have

$$\begin{aligned} \langle Q^2 \rangle + \langle U^2 \rangle + \langle V^2 \rangle &= \langle I^2 \rangle - \\ & 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1 \mathcal{E}_2 e^{i(\phi_2-\phi_1)} \rangle \langle \mathcal{E}_1 \mathcal{E}_2 e^{-i(\phi_2-\phi_1)} \rangle \\ &= \langle I^2 \rangle - \\ 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2-\phi_1) \rangle &+ \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \sin^2(\phi_2-\phi_1) \rangle), \end{aligned} \tag{47}$$

and, by Schwartz' inequality ($\langle ab \rangle \geq \langle a \rangle \langle b \rangle$),

$$I^2 \geq Q^2 + U^2 + V^2. \tag{48}$$

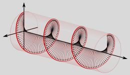
2.4- Quasi-monochromatic waves

- For a wave with a single and constant eliptical polarization state, then the equality holds in Eq. 48.
- On the other hand, for a completely unpolarized wave, $Q = U = V = 0$.
- The Stokes parameters are additive. Proof: consider a sum of N different waves

$$\vec{E} = \sum_{k=1}^{N} \vec{E}^k = \sum (\hat{\epsilon}_1 E_1^k + \hat{\epsilon}_2 E_2^k) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \tag{49}$$

Because each $E_i^k(t)$ is statistically independent, $\langle E_i^k E_j^{l*} \rangle = \delta_{kl} \langle E_i^k E_j^{k*} \rangle$, and

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \sum_k \begin{pmatrix} I_k \\ Q_k \\ U_k \\ V_k \end{pmatrix}. \tag{50}$$

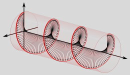


2.4- Quasi-monochromatic waves

- We can therefore decompose an arbitrary set of Stokes parameters in

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \overbrace{\begin{pmatrix} I - \sqrt{Q^2+U^2+V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\text{unpol}} + \overbrace{\begin{pmatrix} \sqrt{Q^2+U^2+V^2} \\ Q \\ U \\ V \end{pmatrix}}^{\text{pol}} . \qquad (51)$$

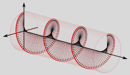
- The first term ‘unpol’ is completely unpolarized since $Q = U = V = 0$, while the second term ‘pol’ is completely polarized since it satisfies $I^2 = Q^2 + U^2 + V^2$ (Eq. 43).



2.4- Quasi-monochromatic waves

- The total polarized intensity of a wave train is thus be $I^{\text{pol}} = \sqrt{Q^2 + U^2 + V^2}$.
- We define the polarization fraction as

$$\Pi = \frac{I^{\text{pol}}}{I}. \tag{52}$$



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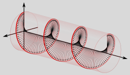

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3. **Wave propagation in a medium**
   - Constitutive equations
   - Kramers-Kronig relations
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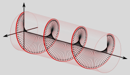


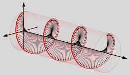
3.1 Constitutive equations

- Each monochromatic component of the field $\vec{E}, \vec{B}$ must fulfill the following constitutive relations:

$$\begin{array}{rcl} \vec{P} = \epsilon_\circ \chi \vec{E} & \longrightarrow & \vec{P}(\omega) = \epsilon_\circ \chi(\omega) \vec{E}(\omega), \\ \vec{B} = \mu \vec{H} & \longrightarrow & \vec{B}(\omega) = \mu(\omega) \vec{H}(\omega), \\ \vec{J} = \sigma \vec{E} & \longrightarrow & \vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega), \end{array} \tag{53}$$

in which we have added Ohm's law.

- We note that $\chi(-\omega) = \chi^*(\omega)$, so that $\chi(t) = \frac{1}{2\pi} \int d\omega \chi(\omega) \exp(-i\omega t)$ be real (and similarly for μ and σ).



3.1 Constitutive equations

- The Fourier convolution theorem states that if $X(\omega) = Y(\omega)Z(\omega)$, then

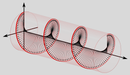
$$X(t) = \int_{-\infty}^{\infty} Y(t - t')Z(t')dt', \qquad (54)$$

where $Y(t) = \frac{1}{2\pi}\int d\omega Y(\omega)\exp(-i\omega t)$, etc..

- Applying the convolution theorem to χ (for example),

$$P(t) = \int_{-\infty}^{\infty} G(t - t')E(t')dt', \text{ with}$$

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon_\circ \chi(\omega) e^{-i\omega t} d\omega. \qquad (55)$$



3.1 Constitutive equations

- We see that $P(t)$ depends on the history of $\vec{E}(t')$, which bears physical sense only in the past, for $t' < t$, so $G(t) = 0$ if $t < 0$. We will use this property in the next section.
- This time we write the monochromatic wave as

$$\vec{E}(t) = \vec{A}\cos(\omega_\circ t) + \vec{B}\sin(\omega_\circ t) = \Re(\vec{E}_c(t)), \qquad (56)$$

with $\vec{E}_c = (\vec{A} - i\vec{B})(\cos(\omega_\circ t) + i\sin(\omega_\circ t))$.

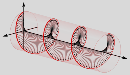
- In the Fourier plane,

$$E(\omega) = \pi\left[(A + iB)\delta(\omega - \omega_\circ) + (A - iB)\delta(\omega + \omega_\circ)\right]. \qquad (57)$$

- We can evaluate

$$P(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon_\circ \chi(\omega) E(\omega) e^{-i\omega t} d\omega$$
$$= \Re\left[\frac{\epsilon_\circ}{2}(A - iB)\chi(\omega_\circ)e^{-i\omega_\circ t}\right] = \Re[P_c(t)], \qquad (58)$$

using $\chi(-\omega) = \chi^*(\omega)$, and where $P_c = \epsilon_\circ \chi(\omega_\circ) E_c(t)$.



3.1 Constitutive equations

- With the spectral decomposition of the constitutive relations we can rewrite the Maxwell equations in their harmonic versions. In the absence of free charges,

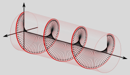
$$\vec{\nabla} \cdot \vec{E}(\omega) = 0, \quad \vec{\nabla} \times \vec{E}(\omega) = -i\omega\mu(\omega)\vec{H}(\omega),$$
$$\vec{\nabla} \cdot \vec{H}(\omega) = 0, \quad \vec{\nabla} \times \vec{H}(\omega) = -i\omega\epsilon(\omega)\vec{E}(\omega), \tag{59}$$

where (tarea)

$$\epsilon(\omega) = \epsilon_\circ(1 + \chi(\omega)) + i\frac{\sigma(\omega)}{\omega}. \tag{60}$$

- Note that both susceptibility and conductivity contribute to the imaginary part of ϵ :

$$\Im(\epsilon) = \epsilon_\circ\Im(\chi) + \Re(\sigma/\omega). \tag{61}$$



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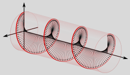


3.2 Kramers-Kronig relations

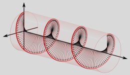
- From physical considerations we can anticipate that the induced $P(t)$ depends on the history of the applied field, or

$$\vec{P}(t) = \int_{-\infty}^{\infty} G(t,t')\vec{E}(t')dt' \qquad (62)$$

(note difference with Eq. 55).

- Let's assume that $\vec{E} = \delta(t - t_\circ)\vec{E}_\circ$. Then $\vec{P}(t) = G(t, t_\circ)\vec{E}_\circ$, and G is the polarization resulting from a delta-unitary electric field.
- If the properties of the medium do not change in time, $G(t, t_\circ) = G(t - t_\circ)$, and we recover Eq. 55.
- Causality requires that $G(\tau) = 0$ if $\tau < 0$, so

$$\epsilon_\circ \chi(\omega) = \int_0^{\infty} dt G(t) e^{i\omega t}. \qquad (63)$$



3.2 Kramers-Kronig relations

- We extend Eq. 63 to the complex plane with $\tilde{\omega} = \omega_R + i\omega_I$, where $\omega_I > 0$.

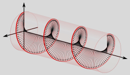
$$\epsilon_\circ \chi(\tilde{\omega}) = \int_0^\infty dt G(t) e^{i\tilde{\omega} t}. \tag{64}$$

- If $\int_0^\infty |G(t)| dt$ converges, so does $\int_0^\infty G(t) e^{i\tilde{\omega} t} dt$, and $\chi(\tilde{\omega})$ is analytical in the superior $\mathbb{C}$ plane ($\omega_I > 0$).
- Therefore $\chi(\tilde{\omega})/(\tilde{\omega} - \omega)$ is analytical except in the pole $\tilde{\omega} = \omega$, where ω is a point along the real axis.
- We can apply the Kramers-Kronig theorem (proof: see Bohren & Huffman, Sec. 2.3.2), which gives

$$i\pi\chi(\omega) = P \int_{-\infty}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega, \tag{65}$$

where P indicates Cauchy's 'principal value'

$$P \int_{-\infty}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega = \lim_{a \to 0} \left(\int_{-\infty}^{\omega - a} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega + \int_{\omega + a}^{\infty} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega \right). \tag{66}$$



3.2 Kramers-Kronig relations

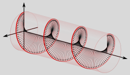
- Using that $\chi^*(\Omega) = \chi(-\Omega)$ we can restrict the integration to $\Omega > 0$, and use $\chi = \chi_R + i\chi_I$ to rewrite Eq. 66:

$$\chi_R(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\Omega\,\chi_I(\Omega)}{\Omega^2 - \omega^2} d\Omega, \tag{67}$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\chi_R(\Omega)}{\Omega^2 - \omega^2} d\Omega. \tag{68}$$

- Similar relationships exists for μ y σ.

Outline



3.3 Monochromatic waves

- We now extend the monochromatic waves to homogeneous media. We inject

$$\vec{E}_c = \vec{E}_\circ e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \text{ and } \vec{H}_c = \vec{H}_\circ e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{69}$$

  into Maxwell's equations.

- Allowing for $\vec{k} \in \mathbb{C}$, $\vec{k} = \underbrace{(k_R + ik_I)}_{k}\,\hat{e}$,

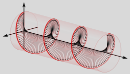
$$\vec{E}_c = \vec{E}_\circ e^{-\vec{k}_I\cdot\vec{x}} e^{i(\vec{k}_R\cdot\vec{x}-\omega t)}. \tag{70}$$

- The harmonic Maxwell equations (Eqs. 59) yield:

$$\begin{array}{cc} \vec{k}\cdot\vec{E}_\circ = 0 & \vec{k}\cdot\vec{H}_\circ(\omega) = 0 \\ \vec{k}\times\vec{E}_\circ = \omega\mu\vec{H}_\circ, & \vec{k}\times\vec{H}_\circ = -\omega\epsilon\vec{E}_\circ. \end{array} \tag{71}$$

- And with $\vec{k}\cdot\vec{k} = \omega^2\epsilon\mu$,

$$k_R^2 - k_I^2 + 2i\vec{k}_I\cdot\vec{k}_R = \omega^2\epsilon\mu \ \ (\text{tarea}). \tag{72}$$



3.3 Monochromatic waves

- For a homogeneous wave (no free charges),

$$\vec{k} = \underbrace{(k_R + ik_I)}_{k}\,\hat{e},$$

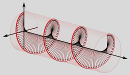
and $k = \omega N/c$, where N is the complex refractive index,

$$N = c\sqrt{\epsilon\mu} = \sqrt{\frac{\epsilon\mu}{\epsilon_\circ\mu_\circ}}.$$

- We set $N = n + i\kappa$, where n and κ are both $\in \mathbb{R}^+$.
- Eq. 70 gives:

$$\vec{E}_c = \vec{E}_\circ e^{-\frac{2\pi}{\lambda}\kappa z} e^{i(\frac{2\pi nz}{\lambda} - i\omega t)}. \qquad (73)$$

- $\Rightarrow$ the imaginary part of N corresponds to absorption.



3.3 Monochromatic waves

- We can apply the Kramers-Kronig relations to $(N(\omega) - 1)$ (the -1 is motivated by $\lim_{\omega\to\infty} N(\omega) = 1$):

$$\begin{aligned} n(\omega) - 1 &= \frac{2}{\pi} P \int_0^\infty \frac{\Omega\,\kappa(\Omega)}{\Omega^2 - \omega^2} d\Omega \\ \kappa(\omega) &= -\frac{2\omega}{\pi} P \int_0^\infty \frac{n(\Omega)}{\Omega^2 - \omega^2} d\Omega \end{aligned} \qquad (74)$$

- We see that the absorption in a medium is also related to the real refractive index.