Part I
Radiative Transfer

Contents

1 Specific Intensity

1.1 Definition of specific intensity

- In geometrical optics we approximate the transport of radiative energy along straight lines, or 'rays', in direction $\hat{k}$.
- If dE is the amount of radiative energy crossing an area $d\vec{A} = dA\hat{n}$ in the direction $\hat{k}$, within a solid angle $d\Omega$, and with a frequency range within $[\nu, \nu + d\nu]$, then the monochromatic specific intensity $I_\nu(\hat{k}, \vec{x}, t)$ is defined by

$$dE = I_\nu(\hat{k}, \vec{x}, t)\,\hat{k}\cdot\hat{n}\,dA\,d\Omega\,d\nu\,dt. \tag{1}$$

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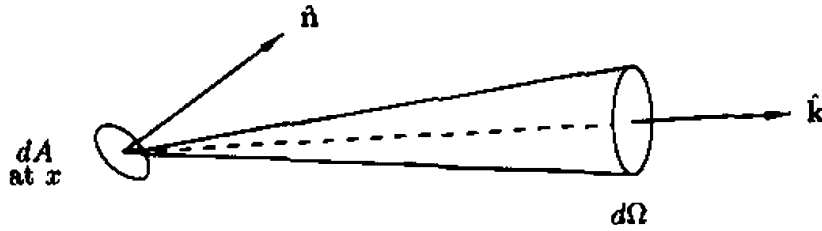
- Geometrical optics can only be applied to cases where the wavelength λ is much smaller than the scale of structure in the propagation medium, 'a'. In other words, it cannot be used to describe the interaction of optical light with atoms, and neither can it be used to describe diffraction phenomena which become evident when $\lambda \sim a$.
- To see this we follow Rybicky & Lightman (Sec. 2.6). Because of the uncertainty principle, for a ray with $\hat{k} = \hat{z}$,

$$dx dp_x dy dp_y = p^2 dA d\Omega \gtrsim h^2, \qquad (2)$$

and

$$dA d\Omega \gtrsim \lambda^2. \qquad (3)$$

- We see that if $dA \sim \lambda^2$, then $d\Omega \sim 1$, and there is no constraint on direction. Similar constraints apply to time, with $d\nu dt \gtrsim 1$.
- Thus the transfer theory described below is applicable when $\lambda \ll a$.

- In the absence of interaction with matter, $I(\hat{k}, \vec{x})$ is constant for any point $\vec{x}$ along the ray $\hat{k}$. Proof: see Shu Vol. I.
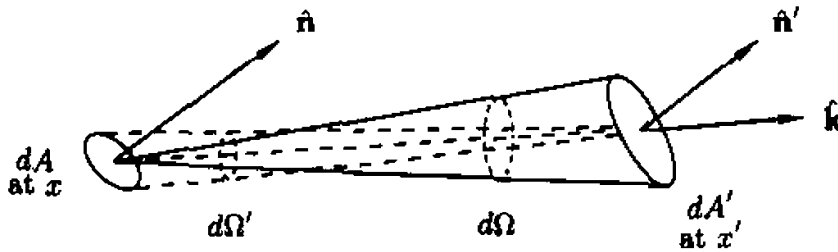
1.2   Moments of specific intensity

- The mean intensity at $\vec{x}$ is simply

$$J_\nu(\vec{x}) = \frac{1}{4\pi} \int d\Omega I_\nu(\hat{k}, \vec{x}). \tag{4}$$

  In spherical coordinates, with $\mu = \cos(\theta)$,

$$J_\nu = \frac{1}{4\pi} \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^{1} d\mu \, I_\nu(\mu, \Phi) \tag{5}$$

- The monochromatic radiative energy density is

$$u_\nu = \frac{4\pi}{c} J_\nu. \tag{6}$$

  and the bolometric radiative energy density is

$$u = \int d\nu u_\nu. \tag{7}$$

- The net flux in the direction $\hat{n}$ and at point $\vec{x}$ is

$$F_\nu(\hat{n}) = \int d\Omega I_\nu(\hat{k}, \vec{x}) \cos(\theta), \tag{8}$$

  where θ is the angle between $\hat{k}$ and $\hat{n}$. With $\mu = \cos(\theta)$,

$$F_\nu(\hat{n}) = \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^{1} d\mu \, \mu I_\nu(\mu, \Phi). \tag{9}$$

- Given that the photon momentum is ϵ/c, with $\epsilon = h\nu$, the radiation pressure in direction $\hat{n}$ is

$$p(\hat{n}) = \frac{2}{c} \int d\Omega \cos^2(\theta) I_\nu(\hat{k}, \vec{x}), \tag{10}$$

  or, in terms of $\mu = \cos(\theta)$,

$$p(\hat{n}) = \frac{2}{c} \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^{1} d\mu \, \mu^2 I_\nu(\mu, \Phi). \tag{11}$$

- Problem: consider a sphere with radius R which emits a uniform specific intensity field from its surface, B. Proove that the flux in direction $\hat{r}$ a distance r away from the center of the sphere is

$$F = \pi B \left(\frac{R}{r} \right)^2. \tag{12}$$

  The flux at the surface of the sphere is thus $F = \pi B$.

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2 Transfer equation

- In the presence of matter the spec. int. field may change with position (and time),

$$\frac{\partial I_\nu}{c\partial t} + \hat{k} \cdot \vec{\nabla} I_\nu = \text{sources} - \text{sinks}. \tag{13}$$

- The sources correspond to the emission of radiative energy by matter, and to reflection in the direction $\hat{k}$.
- The sinks correspond to absorption of radiative energy by matter, or by reflection out of the direction $\hat{j}$.
- Reflected light is also called 'scattered' light.

2.1 Emission

- In the presence of mater the specific intensity field may change with position. A volume element $dV = ds\, dA$ emits an energy
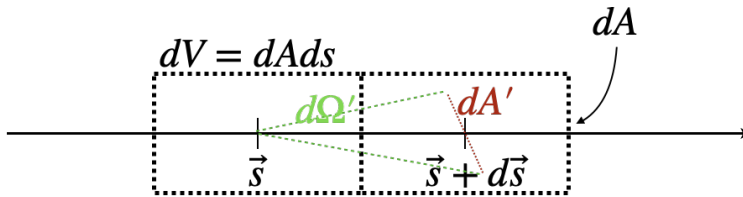
$$dE = j_\nu d\nu dV\, dt d\Omega, \tag{14}$$

towards the direction of $d\Omega$, where $j_\nu(\hat{k})$ is the emission coefficient.
- From the definition of specific intensity,

$$dE = dI_\nu(\hat{k}', \vec{s}, t)\, \hat{k}' \cdot \hat{n}' dA'\, d\Omega'\, d\nu\, dt, \tag{15}$$

for any surface $d\vec{A}'$ centered on $\vec{s}$. If we choose $d\vec{A} = d\vec{A}'$, $\hat{k} = \hat{s}$ and $d\Omega = d\Omega'$ then $\hat{k} \cdot \hat{n} = 1$. In the absence of absorption I_ν is constant along $\hat{s}$, so the increase in I_ν due to the contribution from dV at $\vec{s}$ is

$$dI_\nu = j_\nu ds. \tag{16}$$

2.2 Absorption

- For pure absorption, without scattering or emission, and if the absorption is caused by infinitely small particles that do not shadow each other inside a volume $dV = dsdA$, we expect

$$dI_\nu = -\alpha_\nu I_\nu ds, \tag{17}$$

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- The absorption coefficient α_ν can be connected to the effective cross-sectional area σ_ν of the absorbers. The total energy absorbed in volume dV is

$$dE = -I_\nu n \sigma_\nu dV \, d\Omega dt ds, \qquad (18)$$

where the minus sign accounts for dE being removed from the radiation field.
- The difference in specific intensity is given by

$$dE = dI_\nu(\hat{k}, \vec{x}, t) \, dA \, d\Omega \, d\nu \, dt, \qquad (19)$$

so

$$\alpha_\nu = n \, \sigma_\nu. \qquad (20)$$

2.3   Scattering

- Reflection of photons, rather than absorption, will also decrease the specific intensity along a ray $\hat{k}$. In analogy with pure absorption, the corresponding decrease in specific intensity will be

$$dI_\nu = -\alpha_\nu^{\text{sca}} I_\nu \, ds, \qquad (21)$$

with $\hat{s} = \hat{k}$.
- Photons coming from all directions may also be reflected into the ray $\hat{k}$. To describe the probability of reflection from direction $\hat{k}'$ into direction $\hat{k}$ we use the phase function $\Phi_\nu(\hat{k}, \hat{k}')$ (also called the 'scattering probability density'),

$$dI_\nu = \alpha_\nu^{\text{sca}} \oint \Phi_\nu(\hat{k}, \hat{k}') I_\nu(\hat{k}') d\Omega' \, ds. \qquad (22)$$

The phase function is normalized and symmetric:

$$\oint \Phi_\nu(\hat{k}, \hat{k}') d\Omega' = 1 = \oint \Phi_\nu(\hat{k}', \hat{k}) d\Omega. \qquad (23)$$

- The full transfer equation in a continuous medium is thus:

$$\frac{dI_\nu}{ds} = j_\nu^{\text{abs}} - \alpha_\nu^{\text{abs}} I_\nu - \alpha_\nu^{\text{sca}} I_\nu + \underbrace{\alpha_\nu^{\text{sca}} \oint \Phi_\nu(\hat{k}, \hat{k}') I_\nu(\hat{k}') d\Omega'}_{j_\nu^{\text{scat}}}, \qquad (24)$$

with $\hat{k} = \hat{ds}$, and where we have added the superscript 'abs' to the emission coefficient in the absence of scattering.

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- In thermodynamical equilibrium, $I_\nu(\hat{k}, \vec{x}) = B_\nu(T)$ and since the temperature field must be uniform, $\frac{dI_\nu}{ds} = 0$. Since the phase function is normalized, we have Kirchoff's law:

$$j_\nu^{\text{abs}} = \alpha_\nu^{\text{abs}} B_\nu. \tag{25}$$

- Note that in Eq. 24, the term highlighted as j_ν^{scat} effectively enters as an emission coefficient, i.e. it is not proportional to I_ν.
- Grouping together terms, Eq. 24 can also be written

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu, \tag{26}$$

with $j_\nu = j_\nu^{\text{abs}} + j_\nu^{\text{sca}}$ and $\alpha_\nu = \alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}$.

2.4 Dicrete medium

- In a discrete medium the coupling of radiation and matter occurs through radiative transitions, each of which bears a line-profile $\phi(\nu)$ which is narrowly centred on the transition frequency $\nu_\circ$.
- The rates of radiative transitions can be described using the Einstein coefficients:
    - A_{ji}: probably of spontaneous radiative decay per unit time per absorber (units for A are s^{-1}).
    - $B_{ij}\bar{J}$: probably of absorption per unit time per absorber, with $\bar{J} = \int d\nu \phi(\nu) J_\nu$ is the frequency-averaged mean specific intensity (so that MKS units of B are $s^{-1}/(\text{Jy sr}^{-1})$).
    - $B_{ji}\bar{J}$: probably of stimulated emission per unit time.

- The line profiles $\phi(\nu)$ are described with Lorentzians, and their width (or the line-widths), $\Delta\nu$, are directly connected to the spontaneous decay rate via the time-energy uncertainty principle, $\Delta E \Delta t \geq \hbar/2$:

$$\Delta t \sim \frac{1}{A_{ji}} \sim \frac{1}{\Delta\nu}. \tag{27}$$

- Detailed balance connects the Einstein coefficients:

$$g_i B_{ij} = g_j B_{ji}, \tag{28}$$

and

$$A_{ji} = \frac{2h\nu_{ji}^3}{c^2} B_{ji}. \tag{29}$$

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- In a discrete and isotropic medium, the RT equation in its compact form (Eq. 26) is also valid, and the emission and absorption coefficients are directly connected to the Einstein coefficients. In this case the phase function is constant, $\Phi_\nu(\hat{k}, \hat{k}') = 1/(4\pi)$.
- The emission coefficient is

$$j_\nu = \frac{h\nu_\circ}{4\pi} n_j A_{ji} \phi(\nu). \tag{30}$$

This coefficient includes what would be called 'scattering' for the lines: a de-excitation following the absorption of the same photon.
- The absorption coefficient is

$$\alpha_\nu = \frac{h\nu_\circ}{4\pi}(n_i B_{ij} - n_j B_{ji})\phi(\nu). \tag{31}$$

Note that stimulated emission acts as negative absorption.

3   Opacity and optical depth

- It is customary to use the opacities κ rather than the absorption coefficients α, with $\alpha = \kappa\rho$, where ρ is the mass density.
- We also define the optical depths along a ray,

$$\tau_\nu^{\mathrm{abs}} = \int ds\, \alpha^{\mathrm{abs}} = \int ds \rho \kappa^{\mathrm{abs}}, \tag{32}$$

and

$$\tau_\nu^{\mathrm{sca}} = \int ds\, \alpha^{\mathrm{sca}} = \int ds \rho \kappa^{\mathrm{sca}}. \tag{33}$$

- The exponential decay in the pure absorption case suggests that $\exp(-\tau_\nu^{\mathrm{abs}})$ is the probability that a photon survives the optical depth τ_ν^{abs} without absorption. This allows us to estimate the mean optical depth travelled by all photons:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu \exp(-\tau_\nu)d\tau_\nu = 1. \tag{34}$$

- The photon mean free path l_ν is the corresponding distance, i.e.

$$l_\nu^{\mathrm{abs}} = \frac{1}{\alpha_\nu^{\mathrm{abs}}} = \frac{1}{n\sigma_\nu^{\mathrm{abs}}}, \tag{35}$$

and similarly for scattering.

3.1   Source Function

- Changing variables to $d\tau_\nu = (\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}})ds$, the full transfer equation can be written as

$$\frac{dI_\nu}{d\tau_\nu} = \underbrace{\frac{j_\nu^{\mathrm{abs}} + j_\nu^{\mathrm{sca}}}{\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}}}}_{S_\nu} - I_\nu, \qquad (36)$$

where we have defined the Source Function S_ν.
- In the absence of scattering, i.e. if $\alpha_\nu^{\mathrm{sca}} = 0$, then $S_\nu = B_\nu$.
- For an isotropic phase function, i.e. if $\Phi_\nu(\hat{k}, \hat{k}') = 1/(4\pi)$, the source function is

$$S_\nu = \frac{j_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}} J_\nu}{\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}}} \qquad (37)$$

4   Solutions of the transfer equation

4.1   The uniform slab

- In the widely used, but seldom exact, case of no scattering, the equation of radiative transfer Eq. 36 is

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu, \qquad (38)$$

where $S_\nu = j_\nu^{\mathrm{abs}}/\alpha_\nu^{\mathrm{abs}}$. for thermal emission , $S_\nu = B_\nu$.
- It is straightforward to show that the solution to Eq. 38 is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau_\nu' e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu'). \qquad (39)$$

- For the uniform slab, with constant S_ν,

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}). \qquad (40)$$

4.2   Random walks

- If a medium is optically thick in scattering we expect a photon to undergo a random walk before its absortion or its escape from the medium. The net displacement is $\vec{R} = \sum_i \vec{s}_i$, and the dispersion in displacement is $\sigma(\vec{R}) \equiv (\langle \|\vec{R}\|^2 \rangle)^{1/2} = \sqrt{N}(\langle \|\vec{s}\|^2 \rangle)^{1/2}$.With $\langle \|\vec{s}\|^2 \rangle)^{1/2} \sim l$, the photon mean free path, we have $\sigma(R) \sim \sqrt{N} l$.

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- The probability of absorption in each step is

$$\epsilon = \frac{\alpha_\nu^{\text{abs}}}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}}. \tag{41}$$

Since all steps are statistically independent, the probability of absorption for N steps is simply $N\epsilon_\nu$, and the average number of steps before absorption is $1/\epsilon_\nu$.
- Therefore, for an infinite medium, $\sigma(R) = l/\sqrt{\epsilon_\nu}$, which is sometimes called the 'thermalization length': if the medium is larger than $\sigma(R)$, then it is said to be effectively thick as most photons will be absorbed within the medium.

4.3 Rosseland approximation

- Next to the idealization of full thermodynamical equilibrium, we can explore the consequences of a 'local' thermodynamical equilibrium, such that the deviations of I_ν from $B_\nu(T)$ are small.
- The transfer equation for isotropic scattering can be written as

$$\frac{dI_\nu}{ds} = -\overbrace{(\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{scat}})}^{\alpha_\nu}(I_\nu - S_\nu), \tag{42}$$

and in steady state, $\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial s} = \hat{s} \cdot \vec{\nabla} I_\nu$.
- For thermal radiative processes, $S_\nu = B_\nu$, so

$$I_\nu = B_\nu - \underbrace{\frac{1}{\alpha_\nu}\frac{\partial I_\nu}{\partial s}}_{\text{1st order correction}}. \tag{43}$$

- Since the first order correction is expected to be small, we approximate $\frac{\partial I_\nu}{\partial s} \approx \frac{\partial B_\nu}{\partial s}$, and

$$I_\nu \approx B_\nu - \frac{1}{\alpha_\nu}\frac{\partial B_\nu}{\partial s}. \tag{44}$$

- For the flux $\vec{F}_\nu = \oint I_\nu \hat{k} d\Omega$ we integrate over solid angle. Concretely,

$$\vec{F}_\nu \cdot \hat{z} \equiv F_\nu = \oint \hat{k} \cdot \hat{z} I_\nu(\Omega) d\Omega = \oint \cos(\theta) I_\nu(\Omega) d\Omega. \tag{45}$$

- For an application we need to rewrite Eq. 44 in terms of $\frac{\partial}{\partial z}$:

$$\frac{\partial B_\nu}{\partial s} = \hat{s} \cdot \vec{\nabla} B_\nu = \mu \frac{\partial B_\nu}{\partial z}, \tag{46}$$

in plane-parallel geometry.
- Direct integration of Eq. 44 then yields

$$F_\nu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu}{\partial z}. \tag{47}$$

- The need for a plane-parallel approximation (required in Eq. 46) can be avoided. We start by defining the angular moments of I_ν:

$$\left\{ \begin{array}{c} cu_\nu \\ \vec{F}_\nu \\ c\mathbf{P}_\nu \end{array} \right\} \equiv \oint \left\{ \begin{array}{c} 1 \\ \hat{k} \\ \hat{k}\hat{k} \end{array} \right\} I_\nu(\hat{k})d\Omega. \qquad (48)$$

- Next we rewrite the RT equation as

$$\hat{k} \cdot \vec{\nabla} I_\nu = j_\nu^{\rm abs} - \alpha_\nu I_\nu + \alpha_\nu^{\rm sca} \oint \Phi(\hat{k}, \hat{k}')I_\nu(\hat{k}')d\Omega'. \qquad (49)$$

- We then evaluate $\oint d\Omega \hat{k} (\mathrm{Eq.}\, 49)$:

$$\vec{\nabla} \cdot \oint \hat{k}\hat{k}I_\nu d\Omega = \underbrace{\oint d\Omega \hat{k}j_\nu^{\rm abs}}_{=0} - \alpha_\nu \vec{F}_\nu +$$

$$\alpha_\nu^{\rm sca} \oint d\Omega' I_\nu(\hat{k}') \underbrace{\oint d\Omega \hat{k} \Phi(\hat{k}, \hat{k}')}_{=0,\ {\rm if}\ \Phi(-\hat{k},\hat{k}')=\Phi(\hat{k},\hat{k}')} . \quad (50)$$

- In Eq. 50 we used forward-backward scattering symmmetry, i.e. that $\Phi(-\hat{k}, \hat{k}') = \Phi(\hat{k}, \hat{k}')$, which is true for atomic processes (but not for large dust grains).
- We thus have

$$c\vec{\nabla} \cdot \mathbf{P} = -\alpha_\nu \vec{F}_\nu. \qquad (51)$$

- We now apply local thermodynamical equilibrium, so that $I_\nu(\vec{x}) \approx B_\nu(T(\vec{x}))$:

$$\mathbf{P}_{ij} \approx \oint d\Omega k_i k_j \frac{B_\nu}{c} = \frac{4\pi}{3c} B_\nu(T)\delta_{ij}. \qquad (52)$$

- Replacing into Eq. 51,

$$F_\nu = \vec{F}_\nu\Big|_z \approx -\frac{4\pi}{3\alpha_\nu} \frac{\partial B_\nu(T)}{\partial z}. \qquad (53)$$

- We integrate over frequency for the total radiative flux:

$$F(z) = \int_0^\infty d\nu F_\nu(z) = -\frac{4\pi}{3} \frac{\partial T}{\partial z} \int_0^\infty d\nu \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T}. \qquad (54)$$

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- We introduce the Rosseland mean absorption coefficient,

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty d\nu \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T}}{\int_0^\infty d\nu \frac{\partial B_\nu}{\partial T}}, \tag{55}$$

and note that $\int_0^\infty d\nu \frac{\partial B_\nu}{\partial T} = \frac{4}{\pi}\sigma T^3$.
- Finally, the flux in LTE is:

$$\vec{F}\cdot\hat{s} = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial s}. \tag{56}$$

- We can also write Eq. 56 as

$$\vec{F}\cdot\hat{s} = -\frac{4\sigma}{3\alpha_R}\frac{\partial T^4}{\partial s}, \tag{57}$$

or

$$\vec{F}\cdot\hat{s} = -\frac{4c}{3\alpha_R}\frac{\partial u}{\partial s}, \tag{58}$$

where u is the total radiative energy density.
- It is interesting to note the similitude between Rosseland's approximation and Fick's law for diffusion:

$$\text{flux} \sim -D\vec{\nabla} \text{ quantity being diffused}, \tag{59}$$

in this case $D = \frac{1}{3}cl$ with $l \equiv 1/\alpha_R$ would be an effective mean free path.

- A problem with Rosseland's approximation is that the diffusion coefficient D diverges as $\rho \to 0$. In order to extend is domain of validity, the 'lambda-controled' diffusion has been proposed:

$$\vec{F}\cdot\hat{s} = -\Lambda\frac{4c}{3\alpha_R}\frac{\partial u}{\partial s}, \tag{60}$$

For instance

$$\Lambda = \frac{2+R}{6+3R+R^2}, \tag{61}$$

with

$$R = \frac{1}{\alpha_R}\frac{\|\vec{\nabla}u\|}{u}, \tag{62}$$

see Casassus et al. (2019, MNRAS, 486L, 58; and references therein) for an application.

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4.4 Eddington approximation

- In plane-parallel geometry the radiation moments introduced in Sec. 1.2 and in Eq. 48 can be summarised as

$$\left\{ \begin{array}{c} J_\nu \\ H_\nu \\ K_\nu \end{array} \right\} \equiv \oint_{-1}^{1} \frac{1}{2} \left\{ \begin{array}{c} 1 \\ \mu \\ \mu^2 \end{array} \right\} I_\nu(\mu) d\mu. \tag{63}$$

- By comparison with Eq. 48, we have $J_\nu = c\frac{u_\nu}{4\pi}$, $H_\nu = \frac{\vec{F}_\nu|_z}{4\pi}$, $K_\nu = c\frac{\mathbf{P}_\nu|_{zz}}{4\pi}$.
- In the treatment that leads to the Eddington approximation, we assume that $I_\nu(\mu)$ is nearly isotropic and we expand to first order in μ:

$$I_\nu = a_\nu + b_\nu \mu, \tag{64}$$

i.e. similar to a dipole term $b\mu$ modifying the monopole term a.

- Direct evaluation of Eq. 63 using Eq. 64 leads to $J_\nu = a_\nu$, $H_\nu = \frac{b_\nu}{3}$ and $K_\nu = \frac{a_\nu}{3}$.
- We therefore have

$$K_\nu = \frac{1}{3} J_\nu, \tag{65}$$

exactly as for thermal radiation, for which the radiation pressure is $p_\nu = \frac{1}{3}u_\nu$, but here in a more general case. Eq. 65 is known as the Eddington approximation.

- The plane-parallel RT equation following from Eq. 42 and Eq. 46 is

$$\mu \frac{\partial I_\nu}{\partial z} = -\alpha_\nu (I_\nu + S_\nu),, \tag{66}$$

or

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = -(I_\nu + S_\nu),. \tag{67}$$

- We now take $\int d\mu$(Eq. 67),

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu, \tag{68}$$

and $\int d\mu\mu$(Eq. 67),

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu. \tag{69}$$

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- Using the Eddington approximation (Eq. 65), we obtain an equation for J_ν as a function of S_ν:

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = J_\nu - S_\nu, \tag{70}$$

- In terms of the albedo $\omega_\nu = 1 - \epsilon_\nu$ (see Eq. 41),

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \epsilon(\nu)(J_\nu - B_\nu), \tag{71}$$

which, if the medium properties are known, is a 2nd-order equation for $J_\nu(\tau_\nu)$. If it is possible to solve Eq. 71 for $J_\nu(\tau_\nu)$, then we have I_ν by integration of Eq. 67.

4.5   Two-stream approximation

- We have seen that the Eddington approximation leads to a 2nd-order equation for J_ν (Eq. 71) which approximately solves the RT problem in plane-parallel geometry. However boundary conditions are required to solve Eq. 71. The two-stream approximations provides a set of such boundary conditions.
- In the two stream approximation, we assume that the whole specific intensity field is restricted to only two directions with $\mu = \pm\frac{1}{\sqrt{3}}$:

$$I_\nu(\mu) = I_\nu^+ \delta\left(\mu - \frac{1}{\sqrt{3}}\right) + I_\nu^- \delta\left(\mu + \frac{1}{\sqrt{3}}\right). \tag{72}$$

- Direct evaluation yields

$$\left\{\begin{array}{c} J_\nu \\ H_\nu \\ K_\nu \end{array}\right\} = \left\{\begin{array}{c} \frac{1}{2}(I^+ + I^-) \\ \frac{1}{2\sqrt{3}}(I^+ - I^-) \\ \frac{1}{6}(I^+ + I^-) \end{array}\right\}. \tag{73}$$

- We see that in the two-stream approximation, $K_\nu = \frac{1}{3}J_\nu$, which is the Eddington approximation. The transfer equation Eq. 67 also applies to the two streams, so taking moments and applying the Eddington approximation also yields Eq. 71. Thus, the two streams also satisfy Eq. 71.
- Using the Eddington approx. in Eq. 69,

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu}, \tag{74}$$

the moments in Eq. 73 can be rewritten as

$$I^+ = J_\nu + \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu}, \tag{75}$$

$$I^- = J_\nu - \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu}. \tag{76}$$

$$\tag{77}$$

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- To finally write the required boundary conditions, we focus on a slab geometry from $\tau_\nu = 0$ to $\tau_\nu = \tau_\circ$. Clearly, no radiation will come from $\tau_\nu = +\infty$, so $I^-(\tau_\circ) = 0$. Likewise, $I^+(0) = 0$. Therefore, the required boundaries are:

$$\frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu} = J_\nu, \text{ at } \tau_\nu = \tau_\circ \tag{78}$$

$$\frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau_\nu} = -J_\nu \text{ at } \tau_\nu = 0. \tag{79}$$

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