

Part III

Radiation

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1 Green function for the wave equation

- In order to determine $\vec{A}(\vec{x}, t)$ and $\Phi(\vec{x}, t)$, we need to solve the wave equation with source terms. For a generic field $\Psi(\vec{x}, t)$,

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t). \quad (1)$$

- It is convenient to use the Fourier time-domain,

$$\psi(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(\vec{x}, \omega) e^{-i\omega t} d\omega, \quad (2)$$

whose inverse is

$$\psi(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \psi(\vec{x}, t) e^{i\omega t} dt. \quad (3)$$

- Injecting Ec. 2 in Ec. 1, we reach the Helmholtz equation:

$$(\nabla^2 + k^2)\Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega). \quad (4)$$

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- The Helmholtz equation Eq. 4 is very similar to the Poisson equation, and we can anticipate the use of similar machinery in its solution. The Green function satisfies

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}, \vec{x}'). \quad (5)$$

- Changing coordinates to a system with origin in \vec{x}' , we see that Eq. 5 has spherical symmetry, and $G_k(\vec{x}, \vec{x}') = G_k(R)$, with $R = |\vec{R}|$ and $\vec{R} = \vec{x} - \vec{x}'$
- Eq. 5 can thus be written as

$$\frac{1}{R} \frac{d^2}{dR^2}(R G_k) + k^2 G_k = -4\pi\delta(\vec{R}). \quad (6)$$

- If $R \neq 0$, the solution to Eq. 6 is $R G_k = A e^{ikR} + B e^{-ikR}$, where the constants A and B do not depend on k . In order to determine these constants, we use the case $k = 0$, i.e. Poisson, whose solution is $G_{k=0}(R) = 1/R$, $\rightarrow A + B = 1$.
- Thus the general solution to Eq. 6 is

$$G_k(R) = A G_k^+(R) + B G_k^-(R), \text{ with } G_k^\pm = \frac{e^{\pm ikR}}{R} \quad \text{and } A + B = 1. \quad (7)$$

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- The values of A and B depend on the initial conditions, i.e. on the boundary conditions in time. To see this, we return to the time domain and we generalize Eq. 5:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2} \right) G^\pm(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \quad (8)$$

- Now, returning to the frequency domain ω , we generalize Eq. 5 to

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}'; t') = -4\pi\delta(\vec{x}, \vec{x}')e^{i\omega t'}, \quad (9)$$

with solution $G_k^\pm(R)e^{i\omega t'}$.

- To return once more to the time domain, we use Eq. 2, and

$$G^\pm(R; t, t') = G^\pm(R, \tau) = \frac{1}{2\pi} \int \frac{e^{\pm ikR - i\omega\tau}}{R} d\omega,$$

where $\tau = t - t'$.

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- For a non-dispersive medium (one with $\omega/k = c$), we reach

$$G^\pm(\vec{x}, t; \vec{x}', t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x} - \vec{x}'|}{c}\right]\right)}{|\vec{x} - \vec{x}'|} \quad (10)$$

- We apply this Green function to write the generic solutions of Ec. 1:

$$\Psi^\pm(\vec{x}, t) = \int d^3x' dt' G^\pm(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'). \quad (11)$$

- The $+$ case corresponds to the retarded solution, with an entry or an initial condition ψ_{in} (valid before the sources f are activated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t < 0$):

$$\Psi^+(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t) + \int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (12)$$

where we see that if $t < 0$, there is no \vec{x} for any given \vec{x}' such that $\left[t - \frac{|\vec{x} - \vec{x}'|}{c}\right] > 0$. Hence if $t < 0$, $\int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t)$.

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- Instead the $-$ case corresponds to the anticipated solution, with an exit condition Ψ_{out} (after the sources f are deactivated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t > 0$),

$$\Psi^-(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t) + \int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (13)$$

where we see that if $t > 0$, there is no \vec{x} for any given \vec{x}' such that $\left[t + \frac{|\vec{x} - \vec{x}'|}{c}\right] < 0$. Hence if $t > 0$, $\int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t)$.

- In general we use the retarded solution Ψ^+ .

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2 Retarded Potentials

2.1 Application of the Green function to the electrodynamic potentials

- We normally use the retarded solution, with initial condition $\Psi_{\text{in}} = 0$, or, in compact notation,

$$\Psi(\vec{x}, t) = \int d^3x' \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (14)$$

where $[(\dots)]_{\text{re}}$ means to evaluate in $t' = t - |\vec{x} - \vec{x}'|/c$.

- Applying to the electrodynamic potentials,

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{[\rho(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (15)$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\vec{J}(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}. \quad (16)$$

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2.2 Retarded electromagnetic field

- In order to calculate \vec{E} and \vec{B} , we use $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$.
- Alternatively, we can use the Maxwell equations to reach:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \left(-\vec{\nabla}\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \quad (17)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J}. \quad (18)$$

- Using the Green function, we get

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[-\vec{\nabla}'\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}}, \quad (19)$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[\vec{\nabla}' \times \vec{J} \right]_{\text{ret}}. \quad (20)$$

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- The expressions for the retarded fields Eqs. 19 and 21 can also be written in a form that connects directly with the static expressions (**area**):

$$\begin{aligned} \vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} [\rho(\vec{x}', t')]_{\text{ret}} + \right. \\ \left. \frac{\hat{R}}{cR} \left[\frac{\partial \rho(\vec{x}', t')}{\partial t} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial \vec{J}}{\partial t} \right]_{\text{ret}} \right\} \end{aligned} \quad (21)$$

$$\begin{aligned} \vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \left[\vec{J}(\vec{x}', t') \right]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \right. \\ \left. \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\} \end{aligned} \quad (22)$$

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3 Multipolar Radiation

3.1 Wave zone

- We now consider harmonic sources (the general case can be obtained by superposition of such sources):

$$\begin{aligned}\rho(\vec{x}, t) &= \rho(\vec{x})e^{-i\omega t}, \\ \vec{J}(\vec{x}, t) &= \vec{J}(\vec{x})e^{-i\omega t}.\end{aligned}\quad (23)$$

- We saw that in the presence of sources, the field $\vec{A}(\vec{x}, t)$ generated in vacuum, and without spatial boundaries, is

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta\left(t' + \frac{|\vec{x} - \vec{x}'|}{c} - t\right).$$

- For harmonic sources,

$$\vec{A}(\vec{x}, t) = e^{-i\omega t} \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}, \quad (24)$$

with $k = \omega/c$

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- We then obtain \vec{H} and \vec{E} with

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}, \quad (25)$$

and Faraday's law,

$$\vec{E} = \frac{i}{k} \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{\nabla} \times \vec{H}. \quad (26)$$

- We now consider sources confined inside a region whose maximum extension is d , and that contains the origin. If $d \ll \lambda$, there are 3 regions of interest:
 - The near zone, with $d < r \ll \lambda$, where $e^{ik|\vec{x} - \vec{x}'|} \sim 1$ and we recover the static potentials except for harmonic oscillation, $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$.
 - The intermediate zone with $d \ll r \sim \lambda$.
 - The far zone, with $d \ll r$.

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- In the far zone, with $d \ll r$, $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$, where $\hat{n} = \vec{x}/r \Rightarrow$

$$\lim_{kr \rightarrow \infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') e^{-ik\hat{n} \cdot \vec{x}'} d^3x'. \quad (27)$$

- We see that $\vec{A}(x, t) = \vec{A}(\vec{x})e^{-i\omega t}$ represents a spherical wave travelling outwards.
- In addition, (tarea) using Eqs 25 and 26 we also see that \vec{E} and \vec{H} also form spherical transverse waves (orthogonal to \hat{n}).
- The far zone thus corresponds to the *radiation zone*, also called *wave zone*.

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3.2 Dipolar radiation

- We now use $d \ll \lambda$ to simplify \vec{A} in the wave zone. The integrand in Eq. 27 can be expanded in powers of $-ik\hat{n} \cdot \vec{x}'$, using

$$e^{-ik\hat{n} \cdot \vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n.$$

- Therefore,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}')^n d^3x'. \quad (28)$$

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- For $n = 0$, which is the dominant term in the expansion in $k\hat{n} \cdot \vec{x}'$, we get:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') d^3x'. \quad (29)$$

- Using the continuity equation, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, we have $i\omega\rho = \vec{\nabla} \cdot \vec{J}$.
- Therefore (tarea):

$$\int \vec{J}(\vec{x}') d^3x' = - \int \vec{x}' (\vec{\nabla}' \cdot \vec{J}) d^3x' = -i\omega \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (30)$$

- Finally,

$$\vec{A}(\vec{x}) = \frac{-i\mu_0\omega}{4\pi} \underbrace{\vec{p}}_{\text{electric dipole}} \frac{e^{ikr}}{r}, \quad \text{with } \vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (31)$$

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- We now calculate the \vec{E} and \vec{H} fields:

$$\begin{aligned} \vec{H} &= \frac{ck^2}{4\pi} (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r}, \\ \vec{E} &= \sqrt{\mu_0\epsilon_0} \vec{H} \times \hat{n}, \end{aligned} \quad (32)$$

where we see that electric dipole radiation is linearly polarized.

- The power emitted in direction \hat{n} can be written with $dP = r^2 d\Omega \hat{n} \cdot \vec{S} \Rightarrow$,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2} \Re \left[r^2 \hat{n} \cdot (\vec{E} \times \vec{H}^*) \right], \\ &= \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^4 |\vec{p}|^2 \sin^2(\theta). \end{aligned} \quad (33)$$

- The total power is

$$P = \frac{c^2 k^4}{12\pi} |\vec{p}|^2. \quad (34)$$

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3.3 Magnetic dipole and electric quadrupole radiation

- The next term in the expansion of $e^{-ik\hat{n}\cdot\vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n$ corresponds to $n = 1$.
- Eq. 28 gives:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}') d^3x'. \quad (35)$$

- This term originates *magnetic dipole* and *electric quadrupole* contributions. To see this, we separate the integrand:

$$\vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}') = \underbrace{\frac{1}{2} [(\hat{n} \cdot \vec{x}') \vec{J} + (\hat{n} \cdot \vec{J}) \vec{x}']}_A + \underbrace{\frac{1}{2} (\vec{x}' \times \vec{J}) \times \hat{n}}_B. \quad (36)$$

- We first consider the contribution of part *B* and identify the magnetization $\vec{\mathcal{M}}$,

$$\vec{\mathcal{M}} = \frac{1}{2} (\vec{x} \times \vec{J}). \quad (37)$$

- Then, for the *B* part,

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right), \quad \text{with} \quad (38)$$

$$\vec{m} = \int \vec{\mathcal{M}} d^3x. \quad (39)$$

- In the radiation zone, $kr \gg 1$, we obtain:

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (40)$$

$$\vec{E}(\vec{x}) = -\frac{k^2}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (41)$$

$$\vec{H}(\vec{x}) = -\sqrt{\frac{\epsilon_0}{\mu_0}} (\vec{E} \times \hat{n}). \quad (42)$$

- This contribution is called magnetic dipole radiation.

- For the *A* part in the contribution from $n = 1$, after standard handling we get:

$$A = \frac{i\omega}{2} \int \vec{x}' (\hat{n} \cdot \vec{x}') \rho(\vec{x}') d^3x', \quad (43)$$

which represents order 2 moments of $\rho(\vec{x}')$, i.e. an electric quadrupole contribution, which we will not develop.

4 Radiation from a single charge

4.1 Liénard-Wiechert potentials

- Consider a particle with charge q and trajectory $\vec{r}(t)$, with velocity $\vec{u}(t)$.
- We can apply the retarded solution with source terms $\rho(\vec{x}, t) = q\delta(\vec{x} - \vec{r}(t))$, and $\vec{j}(\vec{x}, t) = \rho(\vec{x}, t)\vec{u}(t)$, and thus obtain the resulting potentials (tarea):

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (44)$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \left[\frac{q\vec{u}}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (45)$$

where $\vec{R}(t') = \vec{x} - \vec{r}(t')$, $R = |\vec{R}|$, $\hat{n}(t') = \frac{\vec{R}}{R}$, and $\vec{\beta}(t') = \frac{\vec{u}(t')}{c}$. These are the Liénard-Wiechert (L.-W.) potentials.

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- We can also calculate the corresponding electromagnetic field (tarea):

$$\vec{E}(\vec{x}, t) = \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{(1 - \beta^2)(\hat{n} - \vec{\beta})}{R^2(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{vel}}} + \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{\left(\hat{n} \times \left((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right)\right)}{cR(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{rad}}} \quad (46)$$

with

$$\vec{B}(\vec{x}, t) = \frac{1}{c} \hat{n} \times \vec{E}(\vec{x}, t). \quad (47)$$

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- We see that far away from the particle, the term labelled E_{rad} will eventually dominate. In fact, for a Fourier component, or for harmonic motion with $\vec{r}(t) \propto \exp(i\omega t)$, we find that (tarea)

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} = \beta \frac{R}{\lambda}, \quad (48)$$

and we see that the radiation term dominates if $R \gg \lambda/\beta$, sometimes also called the *far zone*.

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4.2 Larmor Formula

- In the far zone, with $R \rightarrow \infty$, and in the Galilean limit ($\beta \ll 1$), the power radiated per unit solid angle is (tarea):

$$\frac{dP}{d\Omega} = \frac{\mu_o}{16\pi^2} q^2 a^2 \sin^2(\theta), \quad (49)$$

in which $a = |\dot{\vec{u}}|$ and θ is the angle between \hat{n} and \vec{a} . This is the Larmor formula.

- By applying Eq. 49 to the case of an harmonically oscilating charge with dipole $\vec{p} = q\vec{r}_o \exp(i\omega t)$, we can recover Eq. 33 (tarea).
- We therefore conclude that the *wave zone* as defined in Sec. 3.1 matches the Galilean limit (see the discussion on the dipole approximation in Sec. 3.3 of Rybicki & Lightman).

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4.3 Radiation reaction

- Let's consider a periodic sistem, such that its mechanical state is identical between times t_1 and t_2 .
- Still in the Galilean limit, the total energy radiated by the charge between t_1 and t_2 is

$$W = \frac{\mu_o q^2}{6\pi c} \int_{t_1}^{t_2} a^2 dt. \quad (50)$$

- Energy conservation requires the existence of a 'radiation reaction' force F_{rad} , such that W is extracted from the particle's kinetic energy,

$$\int_{t_1}^{t_2} dt \vec{F}_{\text{rad}} \cdot \vec{u} = -W. \quad (51)$$

- One expression for the radiation reaction is the Abraham-Lorentz formula,

$$\vec{F}_{\text{rad}} = \frac{\mu_o q^2}{6\pi c} \dot{\vec{a}}. \quad (52)$$

We can confirm that Eq. 52 indeeds fullfills Eq. 51 (tarea).

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5 Non-relativistic applications

- Thomson scattering (Rybicki & Lightman Sec. 3.4)
- Harmonically bound particles (Rybicki & Lightman Sec. 3.6)

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