Part IV
Scattering and Diffraction

Contents

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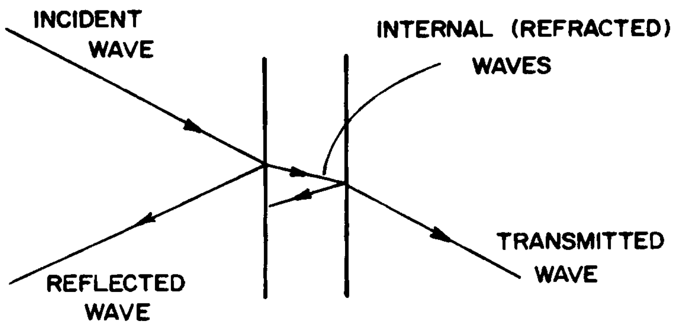
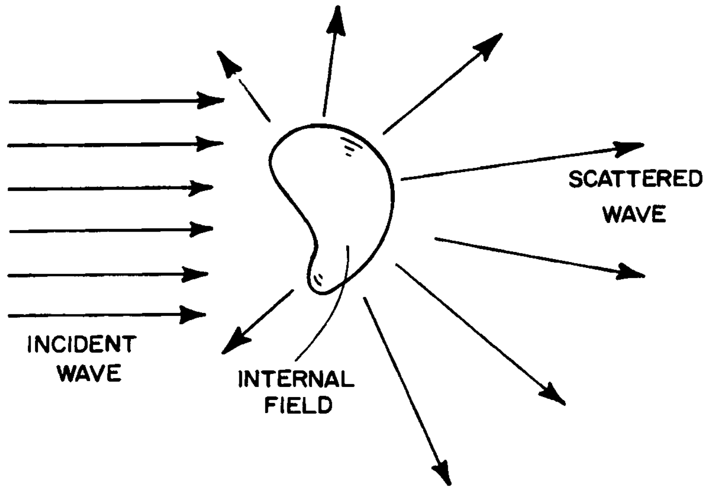
1 Scattering

1.1 General formulation





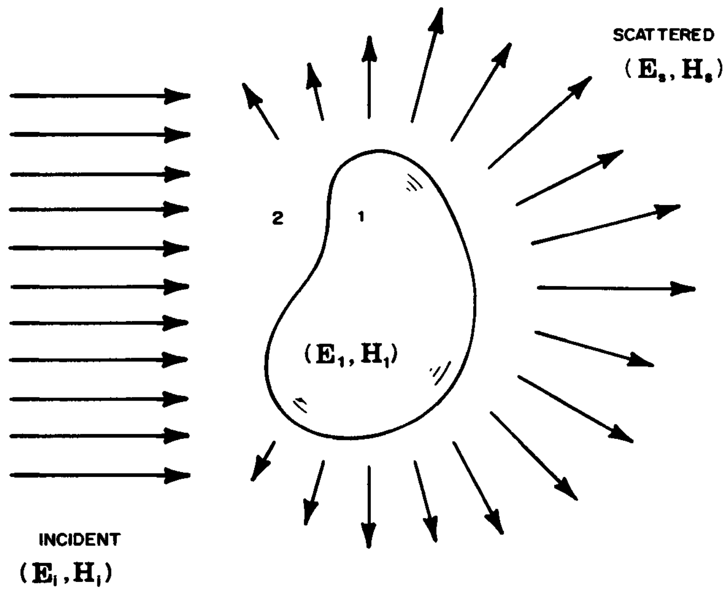
.3

- Let us first consider a single obstacle (or particle), whose maximum dimmension is d.
- The incident wave can be described with

$$\vec{E}_i = \hat{e}_\circ E_\circ e^{ik\hat{n}_\circ \cdot \vec{x}} \quad (1)$$

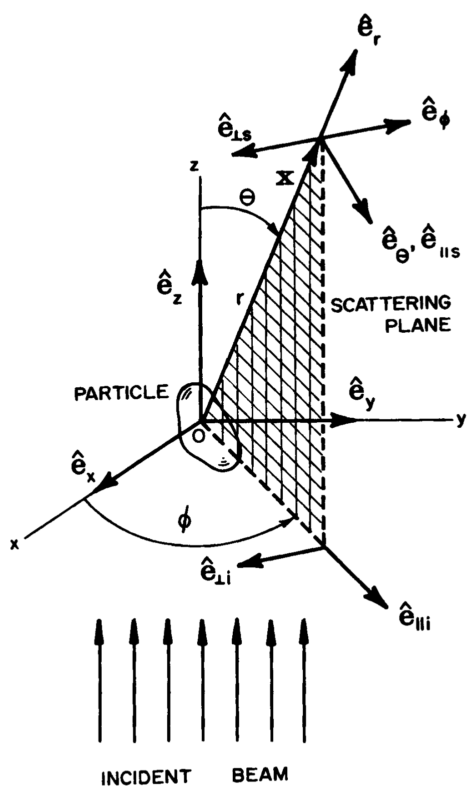
$$\vec{H}_i = \sqrt{\frac{\mu_\circ}{\epsilon_\circ}} \hat{n}_\circ \times \vec{E}_i. \quad (2)$$

- Note that we describe the incident polarization in terms of $\hat{e}_\circ$.
- When interacting with the target, the fields induce electric and magnetic dipoles as in the case of static fields in the 'static zone' (save for the time dependence $\exp(-i\omega t)$).
- The induced dipoles can, in turn, generate electric and magnetic dipole radiation, resulting in the fields $\vec{E}_s$ y $\vec{H}_s$. .4



.5

1.2 Scattering matrix



.6

- In the region outside the target, labelled 2, the fields are given by

$$\vec{E}_2 = \vec{E}_i + \vec{E}_s, \tag{3}$$
$$\vec{H}_2 = \vec{H}_i + \vec{H}_s. \tag{4}$$

- It is convenient to project the fields on the scattering plane. For the incident plane wave,

$$\vec{E}_i = (E_{\circ\|}\hat{e}_{i\|} + E_{\circ\perp}\hat{e}_{i\perp})e^{i(kz-\omega t)} = E_{\|}\hat{e}_{i\|} + E_{\perp}\hat{e}_{i\perp}, \tag{5}$$

  where $\hat{e}_{i\|} \times \hat{e}_{i\perp} = \hat{e}_z$.
- In the wave zone we know that the field emitted by the induced dipoles, i.e. the scattered field, will converge to a transverse wave, i.e. $\|\vec{E}_s\| \propto \frac{e^{ikr}}{r}$, so

$$\vec{E}_s = E_{\|s}\hat{e}_{\|s} + E_{\perp s}\hat{e}_{\perp s}, \tag{6}$$

  with

$$\hat{e}_{\|s} = \hat{e}_\theta, \;\; \hat{e}_{\perp s} = -\hat{e}_\phi, \text{ and } \hat{e}_{\perp s} \times \hat{e}_{\|s} = \hat{e}_r. \tag{7}$$

.7

- Because of the linearity of the Maxwell equations, the scattered fields will be linear combinations of the incident fields.
- We can thus relate the scattered and incident fields in terms of the *amplitude scattering matrix*, with coefficients $\{s_i\}_{i=1}^4$:

$$\begin{pmatrix} E_{\|s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{ikr}}{-ikr} \begin{pmatrix} s_2 & s_3 \\ s_4 & s_1 \end{pmatrix} \begin{pmatrix} E_{\|i} \\ E_{\perp i} \end{pmatrix}. \tag{8}$$

.8

- The time-averaged Poynting vector anywhere outside the target, i.e. in region 2, is

$$\vec{S}_2 = \frac{1}{2}\Re\left[\vec{E}_2 \times \vec{H}_2^*\right] = \vec{S}_i + \vec{S}_s + \vec{S}_{\rm ext}, \tag{9}$$

  where

$$\vec{S}_i = \frac{1}{2}\Re\left[\vec{E}_i \times \vec{H}_i^*\right], \tag{10}$$
$$\vec{S}_s = \frac{1}{2}\Re\left[\vec{E}_s \times \vec{H}_s^*\right], \tag{11}$$
$$\vec{S}_{\rm ext} = \frac{1}{2}\Re\left[\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^*\right]. \tag{12}$$

- The notation "$_{\rm ext}$" anticipates that this term, which corresponds to the interaction between the scattered and incident fields, will cause the *extinction* of the incident specific intensity.

.9

- The Stokes parameters for the scattered fields are similar to the case of plane waves seen in Chap. B, Sec. 2.2.
- For the scattered field we use $\hat{e}_\parallel$ and $\hat{e}_\perp$ rather than $\hat{e}_1$ and $\hat{e}_2$:

$$
\begin{aligned}
I_s &= \langle E_{\parallel s} E^*_{\parallel s} + E_{\perp s} E^*_{\perp s} \rangle & (13)\\
Q_s &= \langle E_{\parallel s} E^*_{\parallel s} - E_{\perp s} E^*_{\perp s} \rangle, & (14)\\
U_s &= \langle E_{\parallel s} E^*_{\perp s} + E_{\perp s} E^*_{\parallel s} \rangle, & (15)\\
V_s &= i\langle E_{\parallel s} E^*_{\perp s} - E_{\perp s} E^*_{\parallel s} \rangle. & (16)
\end{aligned}
$$

.10

- We can now relate the scattered Stokes parameters in terms of the incident Stokes parameters, using the amplitude scattering matrix

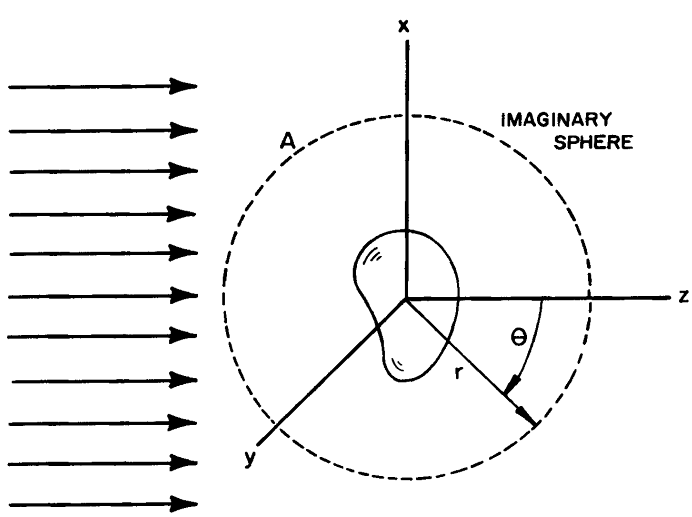
$$
\begin{pmatrix} I_s \\ Q_s \\ U_s \\ V_s \end{pmatrix} = \frac{1}{k^2 r^2} \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix} \begin{pmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{pmatrix}. \quad (17)
$$

- For example, (TAREA):

$$
\begin{aligned}
S_{11} &= \frac{1}{2}(\|s_1\|^2 + \|s_2\|^2 + \|s_3\|^2 + \|s_4\|^2), & (18)\\
S_{12} &= \frac{1}{2}(-\|s_1\|^2 + \|s_2\|^2 - \|s_3\|^2 + \|s_4\|^2), & (19)\\
S_{21} &= \frac{1}{2}(-\|s_1\|^2 + \|s_2\|^2 + \|s_3\|^2 - \|s_4\|^2), & (20)\\
S_{33} &= \frac{1}{2}\Re\left[s_1 s_2^* + s_3 s_4^*\right]. & (21)
\end{aligned}
$$

.11

1.3 Extinction



- Consider a sphere $\mathcal{S}$ centered on a target particle. The net flux of the incident Poynting vector through $\mathcal{S}$ is null, so the flux of the total Poynting vector must correspond to radiative energy produced or absorbed by the particle.

.12

- The total Poynting vector is $\vec{S} = \vec{S}_i + \vec{S}_s + \vec{S}_{\rm ext}$ (see Eqs. 9, 10,11,12), and we write its flux through $\mathcal{S}$ as

$$W_a = -\int_{\mathcal{S}} \vec{S} \cdot \hat{e}_r d\mathcal{S}, \quad (22)$$
$$W_a = W_i - W_s + W_{\rm ext}, \quad (23)$$

where $W_i = -\int \vec{S}_i \cdot \hat{e}_r d\mathcal{S}$, $W_s = +\int \vec{S}_s \cdot \hat{e}_r d\mathcal{S}$ and $W_{\rm ext} = -\int \vec{S}_{\rm ext} \cdot \hat{e}_r d\mathcal{S}$.

- By symmetry W_i=0, so

$$W_{\rm ext} = W_a + W_s, \quad (24)$$

i.e. $W_{\rm ext}$ is the sum of the power absorbed by the particle and that of the scattered radiation.

.13

- We consider a linearly polarized plane wave with $\vec{E}_i \parallel \hat{x}$. In the wave zone, we can write the fields as

$$\vec{E}_s = \frac{e^{ik(r-z)}}{-ikr} \vec{X} E_i, \quad \text{and} \quad (25)$$

$$\vec{H}_s = \frac{k}{\omega\mu} \vec{e}_r \times \vec{E}_s, \quad (26)$$

where $\vec{X}$ is the *vector scattering amplitude*,

$$\vec{X} = (s_2 \cos(\phi) + s_3 \sin(\phi))\hat{e}_{\parallel s} + (s_4 \cos(\phi) + s_1 \sin(\phi))\hat{e}_{\perp s}. \quad (27)$$

Note that $\vec{X}$ is dimensionless, and also depends on θ through the s_i.

.14

- After some calculation (see BH83), in the wave zone ($\lim kr \rightarrow \infty$, tarea),

$$W_{\rm ext} = I_i \frac{4\pi}{k^2} \Re[(\vec{X}.\hat{e}_x)|_{\theta=0}]. \quad (28)$$

- We introduce the extinction cross-section

$$C_{\rm ext} = \frac{W_{\rm ext}}{I_i}, \quad (29)$$

and following Eq. 24, $C_{\rm ext} = C_a + C_s$.

- Using Eqs. 25 and Eqs. 26, we get

$$C_s = \int_{4\pi} \frac{\|\vec{X}\|^2}{k^2} d\Omega. \tag{30}$$

- We identify the differential scattering cross section,

$$\frac{d\sigma_s}{d\Omega}(\theta, \phi) = \frac{\|\vec{X}\|^2}{k^2}, \tag{31}$$

and the scattering phase function

$$\Phi(\theta, \phi) = \frac{1}{C_s} \frac{d\sigma_s}{d\Omega}. \tag{32}$$

.15

- Note that the total cross sections for an assembly of randomly distributed particles is additive (see Sec. 2.2 below). If the particles are spheres, or else are randomly oriented, Φ only depends on θ.
- Another useful quantity is the *asymmetry parameter*,

$$g = \langle \cos(\theta) \rangle = \int_{4\pi} \cos(\theta) \Phi(\theta, \phi) d\Omega. \tag{33}$$

- The cross sections are usually reported in terms of the extinction, scattering and absorption efficiencies,

$$Q_{\mathrm{ext}} = \frac{C_{\mathrm{ext}}}{\Sigma}, \quad Q_{\mathrm{s}} = \frac{C_{\mathrm{s}}}{\Sigma}, \text{ and } Q_{\mathrm{a}} = \frac{C_{\mathrm{a}}}{\Sigma}, \tag{34}$$

where Σ is the projected area of the target in the direction of incidence - i.e. $\Sigma = \pi a^2$ for a sphere with radius a.

.16

- The above cross-sections in Eqs. 29 and 30 were derived for $x-$polarized incident light, i.e. $C_{\mathrm{ext},x}$ and $C_{s,x}$, but are easily extended to $y-$polarized light, $C_{\mathrm{ext},y}$ and $C_{s,y}$.
- For natural light,

$$C_{\mathrm{ext}} = \frac{1}{2}(C_{\mathrm{ext},x} + C_{\mathrm{ext},y}) \text{ and } C_{\mathrm{s}} = \frac{1}{2}(C_{\mathrm{s},x} + C_{\mathrm{s},y}). \tag{35}$$

- If the *scattering volume*, which encompases all targets, includes a continuum of targets with number density n, then we may introduce the *extinction coefficient* which attenuates the incident specific intensity I_ν,

$$\alpha_{\mathrm{ext}} = n C_{\mathrm{ext}}, \tag{36}$$

and

$$dI_\nu = -\alpha_{\mathrm{ext}} I_\nu ds. \tag{37}$$

.17

2 Rayleigh scattering

2.1 Single target

- In the wave zone and in the Rayleigh regime (target $\ll \lambda$), we know from Dipolar Radiation (Chapter C) that the fields in direction $\hat{n}$ are

$$\vec{E}_s = \frac{1}{4\pi\epsilon_\circ} k^2 \frac{e^{ikr}}{r} \left[(\hat{n} \times \vec{p}) \times \hat{n} - \hat{n} \times \frac{\vec{m}}{c} \right] \tag{38}$$

$$\vec{H}_s = \sqrt{\frac{\mu_\circ}{\epsilon_\circ}} \hat{n} \times \vec{E}_s. \tag{39}$$

- We extend the concept of $\frac{dP}{d\Omega}$ to select a polarization state $\hat{e}$ in the scattered wave, and after normalizing by the incident flux, we obtain the differential scattering cross section $\frac{d\sigma}{d\Omega} = \frac{dP}{S_i d\Omega}$:

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_\circ, \hat{e}_\circ) &= r^2 \frac{|\hat{e}^* \cdot \vec{E}_s|^2}{|\hat{e}_\circ^* \cdot \vec{E}_i|^2}, \\ &= \frac{k^4}{(4\pi\epsilon_\circ E_\circ)^2} \left| \hat{e}^* \cdot \vec{p} + (\hat{n} \times \hat{e}^*) \cdot \frac{\vec{m}}{c} \right|^2. \end{aligned} \tag{40}$$

.18

- As an example let's consider the case where the target is a small dielectric sphere, with radius a, $\mu/\mu_\circ = \mu_r = 1$, and with $\epsilon = \epsilon_\circ \epsilon_r(\omega)$.
- In the static zone, where $d \ll r \ll \lambda$, the fields are quasistatic, (tarea)

$$\vec{p} = 4\pi\epsilon_\circ \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) a^3 \vec{E}_i, \tag{41}$$

and there is no magnetic dipole moment.
- The scattering cross section is then, for polarization $\hat{e}$,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 |\hat{e}^* \cdot \hat{e}_\circ|^2. \tag{42}$$

.19

- For natural light, or non-polarized incident radiation, we take the average:

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \left\langle |\hat{e}^* \cdot \hat{e}_\circ|^2 \right\rangle. \tag{43}$$

- In terms of the polarizations parallel and perpendicular to the plane of scattering $(\hat{n}, \hat{n}_\circ)$, for spherical coordinates with $\hat{n}_\circ \parallel \hat{z}$ (TAREA):

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{1}{2} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \cos^2(\theta) \tag{44}$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{2} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \tag{45}$$

- For Stokes I we get

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \frac{1}{2}(1 + \cos^2(\theta)), \tag{46}$$

and a measure of the polarization fraction is $\Pi(\theta) \equiv (\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega})/I = \frac{\sin^2(\theta)}{1+\cos^2(\theta)}$.

.20

2.2 Scattering for N targets

- For a system with N targets, we use the superposition principle,

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_\circ, \hat{e}_\circ) = r^2 \frac{|\hat{e}^* \cdot \sum_{j=1}^{N} \vec{E}_{s,j}|^2}{|\hat{e}_\circ^* \cdot \vec{E}_i|^2}. \tag{47}$$

- In the radiation zone, $|\vec{x} - \vec{x}'| \sim r - \hat{n} \cdot \vec{x}'$,

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_\circ E_\circ)^2} \left| \sum_{j=1}^{N} \left[\hat{e}^* \cdot \vec{p}_j + (\hat{n} \times \hat{e}^*) \cdot \frac{\vec{m}_j}{c} \right] e^{i\vec{q}\cdot\vec{x}_j} \right|^2, \tag{48}$$

where $q = k\hat{n}_\circ - k\hat{n}$ and where the $\{x_j\}$ are the target positions.
- If all targets are identical,

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_1 \mathcal{F}(\vec{q}), \quad \text{where } \mathcal{F}(\vec{q}) = \left| \sum_j e^{i\vec{q}\cdot\vec{x}_j} \right|^2. \tag{49}$$

.21

- If the positions $\vec{x}_j$ are random (TAREA),

$$\langle \mathcal{F}(\vec{q}) \rangle = \left\langle \left| \sum_j e^{i\vec{q}\cdot\vec{x}_j} \right|^2 \right\rangle \approx N, \tag{50}$$

and

$$\frac{d\sigma}{d\Omega} \approx N \frac{k^4}{(4\pi\epsilon_\circ E_\circ)^2} \left| \sum_{j=1}^{N} \left[\hat{e}^* \cdot \vec{p}_j + (\hat{n} \times \hat{e}^*) \cdot \frac{\vec{m}_j}{c} \right] e^{i\vec{q}\cdot\vec{x}_j} \right|^2. \tag{51}$$

.22

- If the targets are regularly ordered, for instance in a cubic network por $N_1 \times N_2 \times N_3$ with spacing a (TAREA),

$$\mathcal{F}(\vec{q}) = N^2 \left[\frac{\sin^2\left(\frac{1}{2}N_1 q_1 a\right) \sin^2\left(\frac{1}{2}N_2 q_2 a\right) \sin^2\left(\frac{1}{2}N_3 q_3 a\right)}{N_1^2 \sin^2\left(\frac{1}{2}q_1 a\right) N_2^2 \sin^2\left(\frac{1}{2}q_2 a\right) N_3^2 \sin^2\left(\frac{1}{2}q_3 a\right)} \right], \tag{52}$$

where $q = q_1\hat{e}_1 + q_2\hat{e}_2 + q_3\hat{e}_3$.

.23

3 Mie theory

3.1 Scattering by a sphere

- Away from the Rayleigh regime, and if the target does not satisfy diffraction boundary conditions (such as for a conductor), then in order to obtain the differential scattering cross section we need to solve the Helmholtz equation for each harmonic component of the fields, subject to interface boundary conditions on the surface of the target ($\mathcal{S}$):

$$\left[\vec{E}_2(\vec{x}) - \vec{E}_1(\vec{x})\right] \times \hat{n} = 0 \text{ and} \tag{53}$$

$$\left[\vec{H}_2(\vec{x}) - \vec{H}_1(\vec{x})\right] \times \hat{n} = 0, \text{ for any } \vec{x} \in \mathcal{S} \tag{54}$$

- The problem is solved by expanding the incident and scattered electric field in a complete set of functions, composed of Legendre polynomials for the θ part, and of spherical Bessel functions for the radial part.

.24

- The solution is expressed in terms of the *size parameter*,

$$x = ka = \frac{2\pi a}{\lambda}, \tag{55}$$

and of $m = k_1/k$, i.e. the real part of the refractive index inside the target.
- The expansion of the scattered fields involves the following coefficients (Eq. 4.53 Bohren & Humman 1998)

$$a_n = \frac{m^2 j_n(mx)[xj_n(x)]' - \mu_1 j_n(x)[mxj_n(mx)]'}{m^2 j_n(mx)[xh_n^{(1)}(x)]' - \mu_1 h_n^{(1)}(x)[mxj_n(mx)]'}, \tag{56}$$

$$b_n = \frac{\mu_1 j_n(mx)[xj_n(x)]' - j_n(x)[mxj_n(mx)]'}{\mu_1 j_n(mx)[xh_n^{(1)}(x)]' - h_n^{(1)}(x)[mxj_n(mx)]'}, \tag{57}$$

where μ_1 is the magnetic permitivity of the target.

.25

- The cross-sections are (Eqs. 4.61 and 4.62 of Bohren & Huffman 1998):

$$C_{\text{sca}} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)\left(|a_n|^2 + |b_n|^2\right) \text{ and} \tag{58}$$

$$C_{\text{ext}} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)\Re\{a_n + b_n\}. \tag{59}$$

.26

- The angle-dependent amplitude scattering matrix is diagonal, $s_3 = s_4 = 0$, and

$$s_1 = \sum \frac{2n+1}{n(n+1)}(a_n\pi_n + b_n\tau_n) \text{ and} \tag{60}$$

$$s_2 = \sum \frac{2n+1}{n(n+1)}(a_n\tau_n + b_n\pi_n), \text{ where} \tag{61}$$

$$\pi_n = \frac{P_n^1}{\sin(\theta)} \text{ and } \tau_n = \frac{dP_n^1}{d\theta}, \tag{62}$$

and where P_n^1 is the Legendre function associated to the corresponding Legendre polynomial

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m P_n(\mu)}{d\mu^m}, \tag{63}$$

with $\mu = \cos(\theta)$.

.27

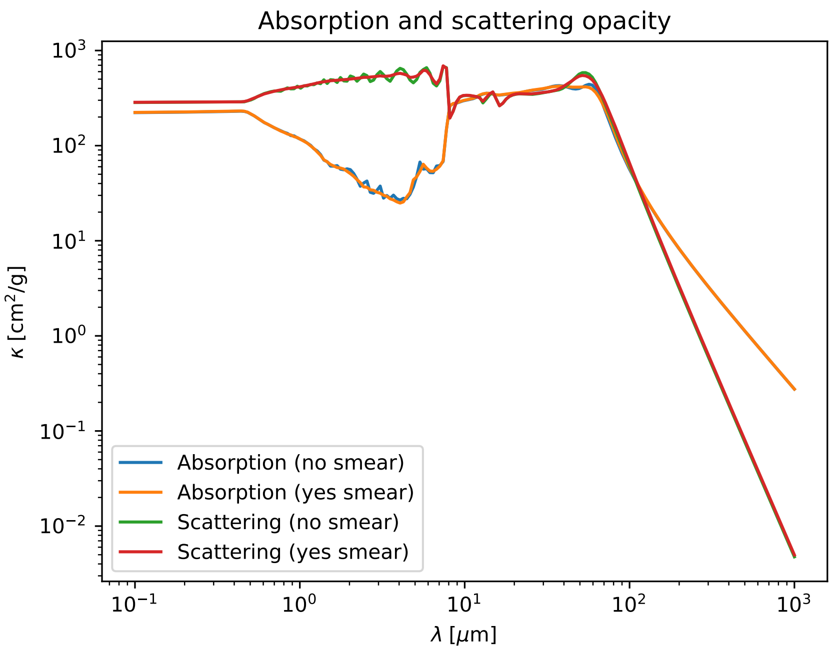
- The scattering phase function for spheres can be obtained from Eqs. 27, 31 and 32:

$$\Phi(\theta) = 2\pi\Phi(\theta,\phi) = 2\pi\frac{s_1^2 + s_2^2}{k^2 C_{\text{sca}}} = 4\pi\frac{S_{11}}{k^2 C_{\text{sca}}}. \tag{64}$$

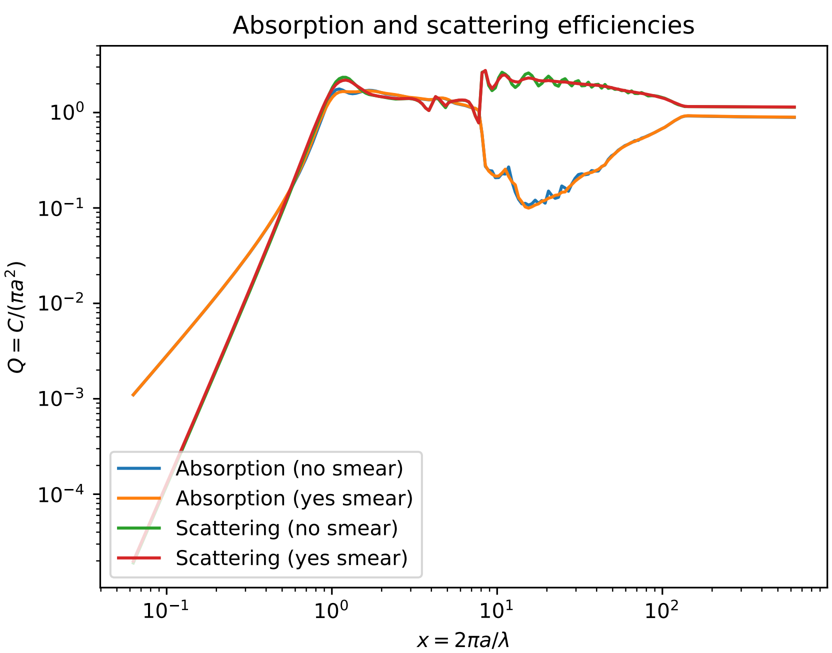
.28

- Standard packages are available to compute the radiative transfer parameters using Mie theory. In what follows, we show the result of the `bhmie.f` code, available here: `https://en.wikipedia.org/wiki/Codes_for_electromagnetic_scattering_by_spheres`. We used the Python transcription and the wrappers from Kees Dullemond, available as part of the RADMC3D Monte-Carlo radiative transfer package: `https://www.ita.uni-heidelberg.de/~dullemond/software/radmc-3d/`.
- The cross sections C are related to the opacities κ by $C = \kappa * m$, where m is the mass of the target sphere.
- Here the phase function Φ is normalized such that $\Phi(\theta,\phi) = 1$ corresponds to isotropic scattering.
- The next 3 plots correspond to $a = 10\mu$m, and used the `pyrmg70` optical constants.

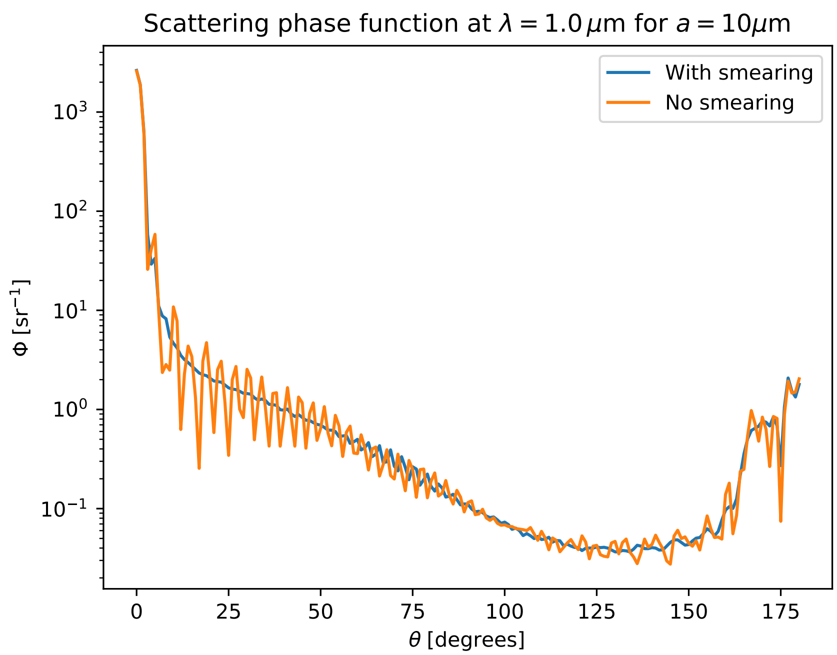
.29


Absorption and scattering opacity
κ [cm²/g]
λ [μm]
Absorption (no smear)
Absorption (yes smear)
Scattering (no smear)
Scattering (yes smear)

.30


Absorption and scattering efficiencies
Q = C/(πa²)
x = 2πa/λ
Absorption (no smear)
Absorption (yes smear)
Scattering (no smear)
Scattering (yes smear)

.31



.32

- TAREA: reproduce the previous 3 plot for a 1 mm-sized sphere composed of pure graphite, and add an extra plot for the grain albedo. Plot the phase function at a wavelength of 1 mm.

.33

4 Diffraction

- The problem of diffraction is similar to scattering, except that we specify the values of the fields at the edges or at the surfaces of the targets.
- Consider a scalar field $\psi(\vec{x}, t)$ which satisfies the wave equation. For a harmonic component, with time dependence $\propto \exp(-i\omega t)$,

$$(\nabla^2 + k^2)\psi(\vec{x}) = 0. \tag{65}$$

- We want to solve the Helmholtz Equation (Eq. 65) for a wave reflected/transmitted at a surface $\mathcal{S}_1$. We close space with another surface, $\mathcal{S}_2$, which we take out to ∞.

.34

- We typically use $\psi = 0$ on $\mathcal{S}_1$, except in possible openings.

- Let us consider the following Green function G_D:

$$G_D(\vec{x}, \vec{x}') = G(\vec{x}, \vec{x}') + F(\vec{x}, \vec{x}'), \tag{66}$$

where

$$(\nabla^2 + k^2)G(\vec{x}, \vec{x}') = -\delta(\vec{x} - \vec{x}'), \tag{67}$$

and

$$(\nabla^2 + k^2)F(\vec{x}, \vec{x}') = 0. \tag{68}$$

- We adjust F so that $G_D(\vec{x}, \vec{x}') = 0$ if $\vec{x} \in \mathcal{S}_1$.

.35

- We will now need the Green identity. For two scalar fields Ψ and Φ,

$$\int_{\mathcal{V}} (\Phi \nabla^2 \Psi + \vec{\nabla}\Phi \cdot \vec{\nabla}\Psi) d^3x = \oint_{\mathcal{A}} \Phi \frac{\partial \Psi}{\partial n} d\mathcal{A}, \tag{69}$$

where

$$\frac{\partial \Psi}{\partial n} \equiv \vec{\nabla}\Psi \cdot \hat{n}.$$

- The Green identity leads to the Green theorem:

$$\int_{\mathcal{V}} (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) d^3x = \oint_{\mathcal{A}} \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) d\mathcal{A}. \tag{70}$$

.36

- The Green Theorem (after an extension to Eq. 65), using the pair G_D and ψ, yields (tarea):

$$\psi(\vec{x}) = \oint_{\mathcal{S}} \left[\psi(\vec{x}')\hat{n}' \cdot \vec{\nabla}' G_D(\vec{x}, \vec{x}') - G_D(\vec{x}, \vec{x}')\hat{n}' \cdot \vec{\nabla}' \psi(\vec{x}') \right] d\mathcal{S}' \tag{71}$$

and using the property that $G_D(\vec{x}, \vec{x}') = 0$ if $\vec{x}' \in \mathcal{S}$,

$$\psi(\vec{x}) = \oint \left[\psi(\vec{x}')\hat{n}' \cdot \vec{\nabla}' G_D(\vec{x}, \vec{x}') \right] d\mathcal{S}'. \tag{72}$$

- Note the absence of the volume integral in the application of Green's theorem that results in Eq. 72, which reflects the absence of sources in the wave equation.

.37

- As an example we focus on the case where $\mathcal{S}_1$ is an infinite plane (at $z = 0$).
- The Green function for the wave equation (Chapter C) is given by:

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi}\frac{e^{ikR}}{R}, \quad \text{with} \quad \vec{R} = \vec{x} - \vec{x}'. \tag{73}$$

- We use the method of images to determine F:

$$F = -\frac{1}{4\pi}\frac{e^{ikR'}}{R'}, \quad \text{with} \quad \vec{R}' = \vec{x} - \vec{x}'', \tag{74}$$

where $\vec{x}''$ is symmetrical to $\vec{x}$ relative to $z = 0$.
- By design the Green function,

$$G_D(\vec{x}, \vec{x}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\right), \tag{75}$$

cancels for $\vec{x}' \in \mathcal{S}_1$.

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- Injecting G_D (Eq. 75) in the Green Theorem (Eq. 72) we get to (tarea):

$$\psi(\vec{x}) = \frac{k}{2\pi i}\oint_{\mathcal{S}_1} \psi(\vec{x}')\frac{\hat{n}' \cdot \vec{R}}{R^2}e^{ikR}\left[1 - \frac{1}{ikR}\right] d\mathcal{S}', \tag{76}$$

where we have used that when $S_2 \to \infty$, $\psi \sim e^{ikR}/R$ on S_2, and $\nabla' G_D \sim 1/R^2$, so that the integrand on S_2 decays faster than $(1/R^2)$.
- If we consider that $\psi(\vec{x}') = 0$ on $\mathcal{S}_1$ except for an opening, in the limit $z \to \infty$, $\frac{\hat{n}' \cdot \vec{R}}{R} \sim 1$,

$$\psi(\vec{x}) = \frac{k}{2\pi i}\int_{\text{opening}} \frac{e^{ikR}}{R}\psi(\vec{x}')d\mathcal{S}', \tag{77}$$

where we recognize the "secondary sources" invoked in Huygens' Principle.

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- For the vectorial case of the electric field difracted by an opening in a plane conductor at $z = 0$, a detailed calculation gives (see Jackson 10.6 and 10.7),

$$\vec{E}(\vec{x}) = \frac{1}{2\pi}\vec{\nabla} \times \int (\hat{n} \times \vec{E}_i)\frac{e^{ikR}}{R}d\mathcal{S}'. \tag{78}$$

- In the region $z \to \infty$, we expect that $\vec{E}$ will be a wave, and if $\vec{E}_i$ is a plane wave,

$$\vec{E}(\vec{x}) \approx \frac{ik}{2\pi}\hat{n} \times (\hat{n}' \times \vec{E}_i)\int \frac{e^{ikR}}{R}d\mathcal{S}'. \tag{79}$$

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