

## Part II

# Electromagnetic wave propagation

## Contents

|           |                                              |          |
|-----------|----------------------------------------------|----------|
| <b>II</b> | <b>Electromagnetic wave propagation</b>      | <b>1</b> |
| II.1      | Electromagnetic waves . . . . .              | 1        |
| II.1.1    | Maxwell Equations . . . . .                  | 1        |
| II.1.2    | Electrodynamic potentials . . . . .          | 1        |
| II.1.3    | Wave equations . . . . .                     | 2        |
| II.1.4    | Poynting's theorem . . . . .                 | 3        |
| II.2      | Wave propagation in vacuum . . . . .         | 5        |
| II.2.1    | Spectral decomposition . . . . .             | 5        |
| II.2.2    | Connection with radiative transfer . . . . . | 6        |
| II.2.3    | Polarization . . . . .                       | 7        |
| II.2.4    | Quasi-monochromatic waves . . . . .          | 8        |
| II.3      | Wave propagation in a medium . . . . .       | 10       |
| II.3.1    | Constitutive equations . . . . .             | 10       |
| II.3.2    | Kramers-Kronig relations . . . . .           | 11       |
| II.3.3    | Monochromatic waves . . . . .                | 12       |

## II.1 Electromagnetic waves

### II.1.1 Maxwell Equations

- In the MKS system (or S.I.), the equations of electrodynamics are, :

$$\vec{\nabla} \cdot \vec{D} = \rho, \quad (\text{II.1})$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (\text{II.2})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (\text{II.3})$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (\text{II.4})$$

- For linear media,  $\vec{D} = \epsilon \vec{E}$  and  $\vec{B} = \mu \vec{H}$ .
- In vacuum,  $\epsilon = \epsilon_0$  and  $\mu = \mu_0$ .

### II.1.2 Electrodynamic potentials

- Since  $\vec{\nabla} \cdot \vec{B} = 0$ , we have

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (\text{II.5})$$

- For  $\vec{E}$ , we use Eq. II.5 and Eq. II.3:

$$\begin{aligned} \vec{\nabla} \times \underbrace{\left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right)}_{\equiv -\vec{\nabla} \Phi} &= 0, \Rightarrow \\ \vec{E} &= -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}. \end{aligned} \quad (\text{II.6})$$

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### II.1.3 Wave equations

- We want to write equations that determine the electrodynamic potentials  $\vec{A}$  and  $\Phi$ .
- Using the Maxwell Equations in vacuum to connect directly with  $\vec{E}$  and  $\vec{B}$ , we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \Phi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = \frac{\rho}{\epsilon_0}, \quad (\text{II.7})$$

$$\begin{aligned} \vec{\nabla} \times \frac{1}{\mu_0} \vec{B} &= \vec{J} + \frac{1}{\epsilon_0} \frac{\partial \vec{E}}{\partial t} \Rightarrow \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \underbrace{\vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right)}_{\text{term for the Lorentz condition}} &= -\mu_0 \vec{J}. \end{aligned} \quad (\text{II.8})$$

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- If the term highlighted in Eq. II.8 is null, which is called the *Lorentz Condition*,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \quad (\text{II.9})$$

then we recover the wave equation for the potentials:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad (\text{II.10})$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}. \quad (\text{II.11})$$

- To fulfill the Lorentz condition, we use the freedom of gauge:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla} \Lambda, \quad (\text{II.12})$$

which leaves invariant  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

- To also preserve  $\vec{E} = -\vec{\nabla} \Phi - \partial \vec{A} / \partial t$ , it is necessary that

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \quad (\text{II.13})$$

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- If  $\vec{A}$  and  $\Phi$  both fulfill the general potential equations (Eqs. II.8 and II.7), but do not fulfill the Lorentz condition, then we can search for  $\Lambda(\vec{x}, t)$  so that  $\vec{A}'$  and  $\Phi'$  do satisfy the Lorentz condition.
- Injecting Eqs. II.12 and II.13 in Eq. II.9, we reach an equation for  $\Lambda(\vec{x}, t)$ :

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0, \quad (\text{II.14})$$

which is essentially a wave equation with a source term, i.e. exactly the type of equations that we will propose solutions for.

.5

- Independently of the Lorentz condition, we can manipulate the Maxwell equations to reach (tarea):

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \left( -\vec{\nabla} \rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \quad (\text{II.15})$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J}, \quad (\text{II.16})$$

which are both wave equations with source terms.

- Away from the sources, i.e. in vacuum, both equations become the homogeneous wave equation.

.6

## II.1.4 Poynting's theorem

- The power exerted by the electromagnetic force  $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$  on a single charge  $q$  with velocity  $\vec{v}$  is  $\vec{v} \cdot \vec{F} = q\vec{v} \cdot \vec{E}$ .
- The power exerted on the charge density distribution  $\rho$  and on the current density distribution  $\vec{J} = \rho\vec{v}$  inside a volume  $d\mathcal{V}$  is thus

$$dP = \vec{J} \cdot \vec{E} d\mathcal{V}$$

- The total power exerted by the  $(\vec{E}, \vec{B})$  field on the charges inside a volume  $\mathcal{V}$  is

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x. \quad (\text{II.17})$$

.7

- We want to connect  $P$  with the energy stored in the fields. Using the Ampère-Maxwell equation (Eq. II.4) we solve for  $\vec{J}$ , and following standard handling (tarea),

$$P = \int_{\mathcal{V}} \left[ -\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \quad (\text{II.18})$$

- Now with the induction law,  $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$  (Eq. II.3),

$$P = \int_{\mathcal{V}} \left[ -\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \quad (\text{II.19})$$

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- Remembering that for a linear medium  $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})$ , and  $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$ , we reach

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x = - \int_{\mathcal{V}} \left[ \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right] d^3x, \quad (\text{II.20})$$

where we recognize

$$u = \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \vec{H}, \quad (\text{II.21})$$

and

$$\vec{S} = \vec{E} \times \vec{H}. \quad (\text{II.22})$$

- For any volume  $\mathcal{V}$ , we conclude that

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}. \quad (\text{II.23})$$

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.9

- In the same way as for energy conservation, Eq. II.23, we can also write the equation for the conservation of linear momentum. Newton's 2nd law for the variation of linear momentum  $\delta \vec{p}_{\text{mec}}$  inside a volume  $\delta \mathcal{V}$  is:

$$\frac{d \delta \vec{p}_{\text{mec}}}{dt} = \rho \vec{E} \delta \mathcal{V} + \rho \vec{v} \times \vec{B} \delta \mathcal{V}. \quad (\text{II.24})$$

- In total,

$$\frac{d \vec{p}_{\text{mec}}}{dt} = \int_{\mathcal{V}} d^3x (\rho \vec{E} + \rho \vec{v} \times \vec{B}). \quad (\text{II.25})$$

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.10

- Using Maxwell's equation to replace  $\rho$  and  $\vec{J}$ , we reach:

$$\frac{d}{dt}(\vec{p}_{\text{mec}} + \vec{p}_{\text{fields}})|_i = \sum_j \int_{\mathcal{V}} d^3x \frac{\partial T_{ij}}{\partial x_j}, \quad (\text{II.26})$$

with the following notations:

$$\vec{p}_{\text{fields}} = \int \epsilon_0 (\vec{E} \times \vec{B}) d^3x = \frac{1}{c^2} \int d^3x \vec{S}, \quad (\text{II.27})$$

which we associate to the momentum in the fields since it fulfills a similar role as  $\vec{p}_{\text{mec}}$ , and

$$T_{ij} = \epsilon_0 \left[ E_i E_j + c^2 B_i B_j - \frac{1}{2} (\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B}) \delta_{ij} \right], \quad (\text{II.28})$$

which is the tensor of electromagnetic tensions.

.11

- For each component  $i$  the integrand of Eq. II.26 (involving  $T_{ij}$ ) can be seen as a divergence, so

$$\frac{d}{dt}(\vec{p}_{\text{mec}} + \vec{p}_{\text{fields}})|_i = \oint_S \sum_j T_{ij} n_j d\mathcal{A}, \quad (\text{II.29})$$

where we recognize a flux integral over the surface bounding the volume  $\mathcal{V}$ .

.12

## II.2 Wave propagation in vacuum

### II.2.1 Spectral decomposition

- In the absence of sources, if we decompose

$$\vec{E}(\vec{x}, t) = \frac{1}{2\pi} \int d\omega \vec{E}(\vec{x}, \omega) e^{i\omega t}, \quad (\text{II.30})$$

the Maxwell equations yield

$$(\nabla^2 + \mu\epsilon\omega^2) \begin{Bmatrix} \vec{E} \\ \vec{B} \end{Bmatrix} = 0. \quad (\text{II.31})$$

- If  $\epsilon$  and  $\mu$  are both real, the solutions are  $e^{\pm i k x}$ , with  $k = \sqrt{\mu\epsilon}\omega$
- We define the phase velocity  $v_\phi = \frac{\omega}{k} = \frac{c}{n}$ , where  $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$  is the refraction index.
- In general,

$$\begin{Bmatrix} E_i \\ B_i \end{Bmatrix} = \frac{1}{2\pi} \int d\omega \begin{Bmatrix} \mathcal{E}_i \\ \mathcal{B}_i \end{Bmatrix} e^{\pm i \vec{k} \cdot \vec{x} - i\omega t}, \quad (\text{II.32})$$

.13

- We recognize d'Alembert's solution for the wave equation,

$$\begin{Bmatrix} E_i \\ B_i \end{Bmatrix} = \frac{1}{2\pi} \int d\omega \begin{Bmatrix} \mathcal{E}_i \\ \mathcal{B}_i \end{Bmatrix} e^{\pm ik(\hat{n} \cdot \vec{x} - v_\phi t)}, \quad (\text{II.33})$$

where each component  $i$  has a form  $f(\hat{n} \cdot \vec{x} - v_\phi t) + g(\hat{n} \cdot \vec{x} + v_\phi t)$ , and where  $\hat{n}$  is the direction of propagation.

- Using Maxwell's equations (tarea),  $\hat{n} \cdot \vec{\mathcal{E}} = 0$ ,  $\hat{n} \cdot \vec{\mathcal{B}} = 0$  and  $\vec{\mathcal{B}} = \frac{n}{c} \hat{n} \times \vec{\mathcal{E}}$ .

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.14

- For harmonic fields it is customary to use complex notation (because of the spectral decomposition), so that  $\vec{S} = \Re(\vec{E}) \times \Re(\vec{H})$ .
- In general for products of the type

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^*b + abe^{-2i\omega t}), \quad (\text{II.34})$$

it is also customary to take time averages  $\langle(\dots)\rangle_T = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\dots) dt$ , and (tarea)

$$\langle\Re(ae^{-i\omega t})\Re(be^{-i\omega t})\rangle = \frac{1}{2}\Re(a^*b). \quad (\text{II.35})$$

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.15

- We therefore have

$$\langle\vec{S}\rangle = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\mathcal{E}|^2\hat{n}. \quad (\text{II.36})$$

And similarly,

$$\langle u \rangle = \frac{1}{4}(\epsilon\vec{E} \cdot \vec{E}^* + \frac{1}{\mu}\vec{B} \cdot \vec{B}^*) = \frac{\epsilon}{2}|\mathcal{E}|^2. \quad (\text{II.37})$$

- Finally,  $\langle\vec{S}\rangle = v_\phi u \hat{n}$ .

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.16

## II.2.2 Connection with radiative transfer

- We can now see that the concept of rays associated to the radiative transfer equation, which describes the transport of radiation in a straight line, is connected to the idea of a plane monochromatic wave with direction of propagation  $\vec{k}$ .
- For a plane wave  $\langle\vec{S}\rangle = v_\phi u \hat{k}$  is the flux of energy in direction  $\hat{k}$ .
- In radiative transfer notation, the flux density in direction  $k_o$  would be

$$F_\nu(\vec{x}) = \int d\Omega I_\nu(\hat{k}, \vec{x}) \hat{k} \cdot \hat{k}_o. \quad (\text{II.38})$$

- Therefore the specific intensity field for a monochromatic plane wave is

$$I_\nu(\hat{k}) = \|\vec{S}_\nu\| \delta(\hat{k} - \hat{k}_o). \quad (\text{II.39})$$

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.17

## II.2.3 Polarization

- In summary, the electric field of a monochromatic wave can be decomposed in two linearly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{e}_1 E_1 + \hat{e}_2 E_2) e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad (\text{II.40})$$

whose total describes, in general, an elliptically polarized wave.

- With a change of vectorial basis to  $\hat{e}_{\pm} = \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$ , we can also decompose  $\vec{E}$  in two circularly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{e}_+ E_+ + \hat{e}_- E_-) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (\text{II.41})$$

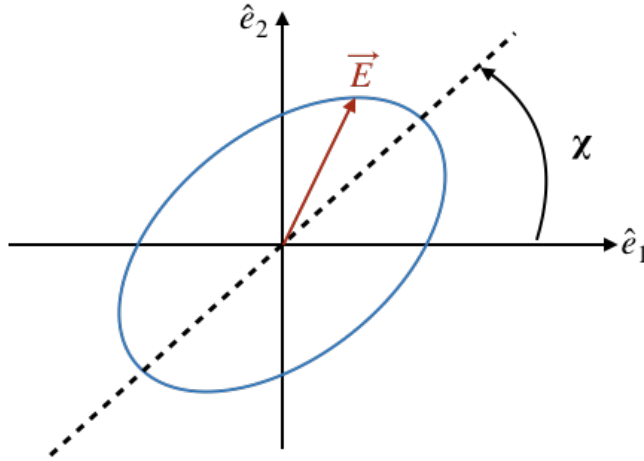
.18

- With the notation

$$\begin{aligned} E_1 &= \mathcal{E}_1 e^{i\phi_1}, & E_2 &= \mathcal{E}_2 e^{i\phi_2}, \\ E_+ &= \mathcal{E}_+ e^{i\phi_+}, & E_- &= \mathcal{E}_- e^{i\phi_-}, \end{aligned}$$

we have

$$\begin{cases} \text{linear polarization : } \phi_2 - \phi_1 = 0. \\ \text{circular polarization : } |\phi_2 - \phi_1| = \frac{\pi}{2} \text{ and } \mathcal{E}_2 = \mathcal{E}_1. \\ \text{the general case is elliptical, with : } \tan(\chi) = \frac{\mathcal{E}_1 \cos(\phi_1)}{\mathcal{E}_2 \cos(\phi_2)}. \end{cases}$$



.19

- It is customary to use the Stokes parameters to characterize the polarization state of **monochromatic light**:

$$\begin{aligned} I &= E_1 E_1^* + E_2 E_2^* = \mathcal{E}_1^2 + \mathcal{E}_2^2, \\ Q &= E_1 E_1^* - E_2 E_2^* = \mathcal{E}_1^2 - \mathcal{E}_2^2, \\ U &= E_1 E_2^* - E_2 E_1^* = 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_2 - \phi_1), \\ V &= i(E_1 E_2^* - E_2 E_1^*) = 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_2 - \phi_1). \end{aligned} \quad (\text{II.42})$$

- We see that Stokes I (the total “radiance”) is  $I \propto |\vec{S}|$ ,  $Q$  and  $U$  measure linear polarization, while  $V$  measure circular polarization. In order to make this obvious it is best to use mental experiments with polarizers that select specific types of polarization (see class).
- For a strictly monochromatic wave, it follows that

$$I^2 = Q^2 + U^2 + V^2. \quad (\text{II.43})$$

.20

## II.2.4 Quasi-monochromatic waves

- In order to obtain  $\vec{E}(\vec{x}, \omega)$ , we need to know  $\vec{E}(t)$  for all  $t$ , since

$$\vec{E}(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{x}, t) e^{i\omega t} dt. \quad (\text{II.44})$$

- So in practice, we treat  $E_1$  and  $E_2$  as random variables, i.e. for a wave in vacuum, described by Eq. II.40,

$$\vec{E}(\vec{x}, t) = (E_1(t)\hat{e}_1 + E_2(t)\hat{e}_2) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \quad (\text{II.45})$$

Alternatively we can also replace the time dependence in Eq. II.45 with a probability density, which itself may depend on time.

- To fix ideas, let's remember that  $\Delta t \Delta \omega = 1$  for Gaussian spectra, where  $\Delta t$  is the ‘coherence time’, and  $\Delta \omega$  is the ‘bandwidth’ of the quasi-monochromatic wave.

.21

- In order to measure the Stokes parameters, we need averages of the kind

$$\langle E_1 E_2^* \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int dt E_1(t) E_2^*(t) dt. \quad (\text{II.46})$$

- We therefore have

$$\begin{aligned} \langle Q^2 \rangle + \langle U^2 \rangle + \langle V^2 \rangle &= \langle I^2 \rangle - \\ 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle + \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \sin^2(\phi_2 - \phi_1) \rangle), \end{aligned} \quad (\text{II.47})$$

and, by Schwartz' inequality ( $\langle \|a\|^2 \rangle \langle \|b\|^2 \rangle \geq \langle a \cdot b \rangle^2$ ),

$$I^2 \geq Q^2 + U^2 + V^2. \quad (\text{II.48})$$

.22

- For a wave with a single and constant elliptical polarization state, the equality holds in Eq. II.48.
- On the other hand, for a completely unpolarized wave,  $Q = U = V = 0$ .
- The Stokes parameters are additive. Proof: consider a sum of  $N$  different waves

$$\vec{E} = \sum_{k=1}^N \vec{E}^k = \sum (\hat{e}_1 E_1^k + \hat{e}_2 E_2^k) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (\text{II.49})$$

Because each  $E_i^k(t)$  is statistically independent,  $\langle E_i^k E_j^{l*} \rangle = \delta_{kl} \langle E_i^k E_j^{k*} \rangle$ , and

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \sum_k \begin{pmatrix} I_k \\ Q_k \\ U_k \\ V_k \end{pmatrix}. \quad (\text{II.50})$$

- We can therefore decompose an arbitrary set of Stokes parameters in

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \overbrace{\begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\text{unpol}} + \overbrace{\begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}}^{\text{pol}}. \quad (\text{II.51})$$

- The first term ‘unpol’ is completely unpolarized since  $Q = U = V = 0$ , while the second term ‘pol’ is completely polarized since it satisfies  $I^2 = Q^2 + U^2 + V^2$  (Eq. II.43).

- The total polarized intensity of a wave train is thus be  $I^{\text{pol}} = \sqrt{Q^2 + U^2 + V^2}$ .
- We define the polarization fraction as

$$\Pi = \frac{I^{\text{pol}}}{I}. \quad (\text{II.52})$$

## II.3 Wave propagation in a medium

### II.3.1 Constitutive equations

- Each monochromatic component of the field  $\vec{E}, \vec{B}$  must fulfill the following constitutive relations:

$$\begin{aligned}\vec{P} &= \epsilon_o \chi \vec{E} &\longrightarrow & \vec{P}(\omega) = \epsilon_o \chi(\omega) \vec{E}(\omega), \\ \vec{B} &= \mu \vec{H} &\longrightarrow & \vec{B}(\omega) = \mu(\omega) \vec{H}(\omega), \\ \vec{J} &= \sigma \vec{E} &\longrightarrow & \vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega),\end{aligned}\tag{II.53}$$

in which we have added Ohm's law.

- We note that Hermitian symmetry,  $\chi(-\omega) = \chi^*(\omega)$ , is required for  $\chi(t) = \frac{1}{2\pi} \int d\omega \chi(\omega) \exp(-i\omega t)$  to be real, and similarly for  $\mu$  and  $\sigma$ .

.26

- The Fourier convolution theorem states that if  $X(\omega) = Y(\omega)Z(\omega)$ , then

$$X(t) = \int_{-\infty}^{\infty} Y(t-t')Z(t')dt',\tag{II.54}$$

where  $Y(t) = \frac{1}{2\pi} \int d\omega Y(\omega) \exp(-i\omega t)$ , etc..

- Applying the convolution theorem to  $\chi$  (for example),

$$\begin{aligned}\vec{P}(t) &= \int_{-\infty}^{\infty} G(t-t')\vec{E}(t')dt', \text{ with} \\ G(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_o \chi(\omega) e^{-i\omega t} d\omega.\end{aligned}\tag{II.55}$$

- We see that  $\vec{P}(t)$  depends on the history of  $\vec{E}(t')$ , which bears physical sense only in the past, for  $t' < t$ , so  $G(t) = 0$  if  $t < 0$ . We will use this property in the next section.

.27

- Note that  $\vec{P}(t)$  is no longer proportional to the applied field  $\vec{E}(t)$ , except for harmonic fields. To see this we write the monochromatic wave as

$$\vec{E}(t) = \vec{A} \cos(\omega_o t) + \vec{B} \sin(\omega_o t) = \Re(\vec{E}_c(t)),\tag{II.56}$$

with  $\vec{E}_c = (\vec{A} - i\vec{B})(\cos(\omega_o t) + i \sin(\omega_o t))$ .

- In the Fourier plane,

$$\vec{E}(\omega) = \pi \left[ (\vec{A} + i\vec{B})\delta(\omega - \omega_o) + (\vec{A} - i\vec{B})\delta(\omega + \omega_o) \right].\tag{II.57}$$

- We can evaluate

$$\begin{aligned}\vec{P}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_o \chi(\omega) \vec{E}(\omega) e^{-i\omega t} d\omega \\ &= \Re \left[ \frac{\epsilon_o}{2} (\vec{A} - i\vec{B}) \chi(\omega_o) e^{-i\omega_o t} \right] = \Re[\vec{P}_c(t)],\end{aligned}\tag{II.58}$$

using  $\chi(-\omega) = \chi^*(\omega)$ , and where

$$\vec{P}_c(t) = \epsilon_o \chi(\omega_o) \vec{E}_c(t).\tag{II.59}$$

.28

- With the spectral decomposition of the constitutive relations we can rewrite the Maxwell equations in their harmonic versions. In the absence of free charges,

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}(\omega) &= 0, & \vec{\nabla} \times \vec{E}(\omega) &= -i\omega\mu(\omega)\vec{H}(\omega), \\ \vec{\nabla} \cdot \vec{H}(\omega) &= 0, & \vec{\nabla} \times \vec{H}(\omega) &= -i\omega\epsilon(\omega)\vec{E}(\omega),\end{aligned}\tag{II.60}$$

where (tarea)

$$\epsilon(\omega) = \epsilon_o(1 + \chi(\omega)) + i\frac{\sigma(\omega)}{\omega}.\tag{II.61}$$

- Note that both susceptibility and conductivity contribute to the imaginary part of  $\epsilon$  :

$$\Im(\epsilon) = \epsilon_o \Im(\chi) + \Re(\sigma/\omega).\tag{II.62}$$

.29

### II.3.2 Kramers-Kronig relations

- From physical considerations we can anticipate that the induced  $P(t)$  depends on the history of the applied field, or

$$\vec{P}(t) = \int_{-\infty}^{\infty} G(t, t') \vec{E}(t') dt' \tag{II.63}$$

(note difference with Eq. II.55).

- Let's assume that  $\vec{E} = \delta(t - t_o) \vec{E}_o$ . Then  $\vec{P}(t) = G(t, t_o) \vec{E}_o$ , and  $G$  is the polarization resulting from a delta-unitary electric field.
- If the properties of the medium do not change in time,  $G(t, t_o) = G(t - t_o)$ , and we recover Eq. II.55.
- Causality requires that  $G(\tau) = 0$  if  $\tau < 0$ , so

$$\epsilon_o \chi(\omega) = \int_0^{\infty} dt G(t) e^{i\omega t}.\tag{II.64}$$

.30

- We extend Eq. II.64 to the complex plane with  $\tilde{\omega} = \omega_R + i\omega_I$ , where  $\omega_I > 0$ .

$$\epsilon_o \chi(\tilde{\omega}) = \int_0^\infty dt G(t) e^{i\tilde{\omega}t}. \quad (\text{II.65})$$

- If  $\int_0^\infty |G(t)| dt$  converges, so does  $\int_0^\infty G(t) e^{i\tilde{\omega}t} dt$ , and  $\chi(\tilde{\omega})$  is analytical in the superior  $\mathbb{C}$  plane ( $\omega_I > 0$ ).
- Therefore  $\chi(\tilde{\omega})/(\tilde{\omega} - \omega)$  is analytical except in the pole  $\tilde{\omega} = \omega$ , where  $\omega$  is a point along the real axis.
- We can apply the Kramers-Kronig theorem (proof: see Bohren & Huffman, Sec. 2.3.2), which gives

$$i\pi\chi(\omega) = P \int_{-\infty}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega, \quad (\text{II.66})$$

where  $P$  indicates Cauchy's 'principal value'

$$P \int_{-\infty}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega = \lim_{a \rightarrow 0} \left( \int_{-\infty}^{\omega-a} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega + \int_{\omega+a}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega \right). \quad (\text{II.67})$$

- Using that  $\chi^*(\Omega) = \chi(-\Omega)$  we can restrict the integration to  $\Omega > 0$ , and use  $\chi = \chi_R + i\chi_I$  to rewrite Eq. II.67:

$$\chi_R(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\Omega \chi_I(\Omega)}{\Omega^2 - \omega^2} d\Omega, \quad (\text{II.68})$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\chi_R(\Omega)}{\Omega^2 - \omega^2} d\Omega. \quad (\text{II.69})$$

- Similar relationships exists for  $\mu$  y  $\sigma$ .

### II.3.3 Monochromatic waves

- We now extend the monochromatic waves to homogeneous media. We inject

$$\vec{E}_c = \vec{E}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \text{ and } \vec{H}_c = \vec{H}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad (\text{II.70})$$

into Maxwell's equations.

- Allowing for  $\vec{k} \in \mathbb{C}$ ,  $\vec{k} = \underbrace{(k_R + ik_I)}_k \hat{e}$ ,

$$\vec{E}_c = \vec{E}_o e^{-\vec{k}_I \cdot \vec{x}} e^{i(\vec{k}_R \cdot \vec{x} - \omega t)}. \quad (\text{II.71})$$

- The harmonic Maxwell equations (Eqs. II.60) yield:

$$\begin{aligned}\vec{k} \cdot \vec{E}_o &= 0 & \vec{k} \cdot \vec{H}_o(\omega) &= 0 \\ \vec{k} \times \vec{E}_o &= \omega\mu\vec{H}_o, & \vec{k} \times \vec{H}_o &= -\omega\epsilon\vec{E}_o.\end{aligned}\quad (\text{II.72})$$

- Eliminating  $\vec{E}_o$  and  $\vec{H}_o$  (for example taking  $\vec{k} \times \vec{k} \times \vec{E}_o$ , as in Bohren & Huffman 2.6), we obtain  $\vec{k} \cdot \vec{k} = \omega^2\epsilon\mu$  (tarea) and

$$k_R^2 - k_I^2 + 2i\vec{k}_I \cdot \vec{k}_R = \omega^2\epsilon\mu \quad (\text{tarea}). \quad (\text{II.73})$$

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.33

- So, for a homogeneous wave (no free charges),

$$\vec{k} = \underbrace{(k_R + ik_I)}_k \hat{e},$$

and  $k = \omega N/c$ , where  $N$  is the complex refractive index,

$$N = c\sqrt{\epsilon\mu} = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}}.$$

- We set  $N = n + i\kappa$ , where  $n$  and  $\kappa$  are both  $\in \mathbb{R}^+$ .
- Eq. II.71 gives:

$$\vec{E}_c = \vec{E}_o e^{-\frac{2\pi}{\lambda}\kappa z} e^{i(\frac{2\pi n z}{\lambda} - i\omega t)}. \quad (\text{II.74})$$

- $\Rightarrow$  the imaginary part of  $N$  corresponds to absorption.

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.34

- We can apply the Kramers-Kronig relations to  $(N(\omega) - 1)$  (the  $-1$  is motivated by  $\lim_{\omega \rightarrow \infty} N(\omega) = 1$ ):

$$\begin{aligned}n(\omega) - 1 &= \frac{2}{\pi} P \int_0^\infty \frac{\Omega \kappa(\Omega)}{\Omega^2 - \omega^2} d\Omega \\ \kappa(\omega) &= -\frac{2\omega}{\pi} P \int_0^\infty \frac{n(\Omega)}{\Omega^2 - \omega^2} d\Omega\end{aligned}\quad (\text{II.75})$$

- We see that the absorption in a medium is also related to the real refractive index.

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.35