

Part III

Radiation

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III.1 General solution of the wave equation

III.1.1 Green function for the wave equation

- This section follows from Jackson 3rd ed, Sec. 6.4.
- In order to determine $\vec{A}(\vec{x}, t)$ and $\Phi(\vec{x}, t)$, we need to solve the wave equation with source terms. For a generic field $\Psi(\vec{x}, t)$,

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t). \quad (\text{III.1})$$

- It is convenient to use the Fourier time-domain,

$$\psi(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(\vec{x}, \omega) e^{-i\omega t} d\omega, \quad (\text{III.2})$$

whose inverse is

$$\psi(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \psi(\vec{x}, t) e^{i\omega t} dt. \quad (\text{III.3})$$

- Injecting Ec. III.2 in Ec. III.1, we reach the Helmholtz equation:

$$(\nabla^2 + k^2)\Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega). \quad (\text{III.4})$$

- The Helmholtz equation Eq. III.4 is very similar to the Poisson equation, and we can anticipate the use of similar machinery in its solution. The Green function satisfies

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}, \vec{x}'). \quad (\text{III.5})$$

- Changing coordinates to a system with origin in $\vec{x}'$, we see that Eq. III.5 has spherical symmetry, and $G_k(\vec{x}, \vec{x}') = G_k(R)$, with $R = |\vec{R}|$ and $\vec{R} = \vec{x} - \vec{x}'$
- Eq. III.5 can thus be written as

$$\frac{1}{R} \frac{d^2}{dR^2}(R G_k) + k^2 G_k = -4\pi\delta(\vec{R}). \quad (\text{III.6})$$

- If $R \neq 0$, the solution to Eq. III.6 is $R G_k = A e^{ikR} + B e^{-ikR}$, where the constants A and B do not depend on k . In order to determine these constants, we use the case $k = 0$, i.e. Poisson, whose solution is $G_{k=0}(R) = 1/R$, $\longrightarrow A + B = 1$.

- Thus the general solution to Eq. III.6 is

$$G_k(R) = A G_k^+(R) + B G_k^-(R), \text{ with } G_k^\pm = \frac{e^{\pm ikR}}{R} \quad (\text{III.7})$$

and $A + B = 1$.

- The values of A and B depend on the initial conditions, i.e. on the boundary conditions in time. To see this, we return to the time domain and we generalize Eq. III.5:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^\pm(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \quad (\text{III.8})$$

- Now, returning to the frequency domain ω , we generalize Eq. III.5 to

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}'; t') = -4\pi\delta(\vec{x}, \vec{x}')e^{i\omega t'}, \quad (\text{III.9})$$

with solution $G_k^\pm(R)e^{i\omega t'}$.

- To return once more to the time domain, we use Eq. III.2, and

$$G^\pm(R; t, t') = G^\pm(R, \tau) = \frac{1}{2\pi} \int \frac{e^{\pm ikR - i\omega\tau}}{R} d\omega,$$

where $\tau = t - t'$.

- For a non-dispersive medium (one with $\omega/k = c$), we reach

$$G^\pm(\vec{x}, t; \vec{x}', t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x} - \vec{x}'|}{c}\right]\right)}{|\vec{x} - \vec{x}'|} \quad (\text{III.10})$$

- We apply this Green function to write the generic solutions of Ec. III.1:

$$\Psi^\pm(\vec{x}, t) = \int d^3x' dt' G^\pm(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'). \quad (\text{III.11})$$

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- The $+$ case corresponds to the retarded solution, with an entry or an initial condition ψ_{in} (valid before the sources f are activated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t < 0$):

$$\Psi^+(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t) + \int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (\text{III.12})$$

where we see that if $t < 0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t - \frac{|\vec{x} - \vec{x}'|}{c}\right] > 0$. Hence if $t < 0$, $\int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t)$.

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- Instead the $-$ case corresponds to the anticipated solution, with an exit condition Ψ_{out} (after the sources f are deactivated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t > 0$),

$$\Psi^-(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t) + \int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (\text{III.13})$$

where we see that if $t > 0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t + \frac{|\vec{x} - \vec{x}'|}{c}\right] < 0$. Hence if $t > 0$, $\int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t)$.

- In general we use the retarded solution Ψ^+ .

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III.1.2 Retarded Potentials

- We normally use the retarded solution, with initial condition $\Psi_{\text{in}} = 0$, or, in compact notation,

$$\Psi(\vec{x}, t) = \int d^3x' \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.14})$$

where $[(\dots)]_{\text{re}}$ means to evaluate in $t' = t - |\vec{x} - \vec{x}'|/c$.

- Applying to the electrodynamic potentials,

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{[\rho(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.15})$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\vec{J}(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}. \quad (\text{III.16})$$

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III.1.3 Retarded electromagnetic field

- In order to calculate $\vec{E}$ and $\vec{B}$, we use $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$.
- Alternatively, we can use the Maxwell equations to reach:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \left(-\vec{\nabla}\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \quad (\text{III.17})$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J}. \quad (\text{III.18})$$

- Using the Green function, we get

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[-\vec{\nabla}'\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}}, \quad (\text{III.19})$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[\vec{\nabla}' \times \vec{J} \right]_{\text{ret}}. \quad (\text{III.20})$$

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- The expressions for the retarded fields Eqs. III.19 and III.21 can also be written in a form that connects directly with the static expressions (Jackson 6.5, **area**):

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} [\rho(\vec{x}', t')]_{\text{ret}} + \frac{\hat{R}}{cR} \left[\frac{\partial \rho(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \right\} \quad (\text{III.21})$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \left[\vec{J}(\vec{x}', t') \right]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\} \quad (\text{III.22})$$

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III.2 Multipolar Radiation

III.2.1 Wave zone

- This section follows Jackson Chap. 9.
- We now consider harmonic sources (the general case can be obtained by superposition of such sources):

$$\begin{aligned} \rho(\vec{x}, t) &= \rho(\vec{x})e^{-i\omega t}, \\ \vec{J}(\vec{x}, t) &= \vec{J}(\vec{x})e^{-i\omega t}. \end{aligned} \quad (\text{III.23})$$

- We saw that in the presence of sources, the field $\vec{A}(\vec{x}, t)$ generated in vacuum, and without spatial boundaries, is

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta \left(t' + \frac{|\vec{x} - \vec{x}'|}{c} - t \right).$$

- For harmonic sources,

$$\vec{A}(\vec{x}, t) = e^{-i\omega t} \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.24})$$

with $k = \omega/c$

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- We then obtain $\vec{H}$ and $\vec{E}$ with

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}, \quad (\text{III.25})$$

and Faraday's law,

$$\vec{E} = \frac{i}{k} \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{\nabla} \times \vec{H}. \quad (\text{III.26})$$

- We now consider sources confined inside a region whose maximum extension is d , and that contains the origin. If $d \ll \lambda$, there are 3 regions of interest:
 - The near zone, with $d < r \ll \lambda$, where $e^{ik|\vec{x} - \vec{x}'|} \sim 1$ and we recover the static potentials except for harmonic oscillation, $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$.
 - The intermediate zone with $d \ll r \sim \lambda$.
 - The far zone, with $d \ll r$.

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- In the far zone, with $d \ll r$, $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$, where $\hat{n} = \vec{x}/r \Rightarrow$

$$\lim_{kr \rightarrow \infty} \vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') e^{-ik\hat{n} \cdot \vec{x}'} d^3x'. \quad (\text{III.27})$$

- We see that $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$ represents a spherical wave travelling outwards.
- In addition, (tarea) using Eqs III.25 and III.26 we also see that $\vec{E}$ and $\vec{H}$ also form spherical transverse waves (orthogonal to $\hat{n}$).
- The far zone thus corresponds to the *radiation zone*, also called *wave zone*.

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III.2.2 Dipolar radiation

- We now use $d \ll \lambda$ to simplify $\vec{A}$ in the wave zone. The integrand in Eq. III.27 can be expanded in powers of $-ik\hat{n} \cdot \vec{x}'$, using

$$e^{-ik\hat{n} \cdot \vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n.$$

- Therefore,

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}')^n d^3x'. \quad (\text{III.28})$$

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- For $n = 0$, which is the dominant term in the expansion in $k\hat{n} \cdot \vec{x}'$, we get:

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') d^3x'. \quad (\text{III.29})$$

- Using the continuity equation, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, we have $i\omega\rho = \vec{\nabla} \cdot \vec{J}$.
- Therefore (tarea):

$$\int \vec{J}(\vec{x}') d^3x' = - \int \vec{x}' (\vec{\nabla}' \cdot \vec{J}) d^3x' = -i\omega \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (\text{III.30})$$

- Finally,

$$\vec{A}(\vec{x}) = \frac{-i\mu_o\omega}{4\pi} \underbrace{\vec{p}}_{\text{electric dipole}} \frac{e^{ikr}}{r}, \quad \text{with } \vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (\text{III.31})$$

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- We now calculate the $\vec{E}$ and $\vec{H}$ fields:

$$\begin{aligned}\vec{H} &= \frac{ck^2}{4\pi} (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r}, \\ \vec{E} &= \sqrt{\frac{\mu_o}{\epsilon_o}} \vec{H} \times \hat{n},\end{aligned}\tag{III.32}$$

where we see that electric dipole radiation is linearly polarized.

- The power emitted in direction $\hat{n}$ can be written with $dP = r^2 d\Omega \hat{n} \cdot \vec{S} \Rightarrow$,

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2} \Re \left[r^2 \hat{n} \cdot (\vec{E} \times \vec{H}^*) \right], \\ &= \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_o}{\epsilon_o}} k^4 |\vec{p}|^2 \sin^2(\theta).\end{aligned}\tag{III.33}$$

- The total power is

$$P = \frac{c^2 k^4}{12\pi} \sqrt{\frac{\mu_o}{\epsilon_o}} |\vec{p}|^2.\tag{III.34}$$

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III.2.3 Magnetic dipole and electric quadrupole radiation

- The next term in the expansion of $e^{-ik\hat{n}\cdot\vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n$ corresponds to $n = 1$.
- Eq. III.28 gives:

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}') d^3x'.\tag{III.35}$$

- This term originates *magnetic dipole* and *electric quadrupole* contributions. To see this, we separate the integrand:

$$\vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}') = \underbrace{\frac{1}{2} \left[(\hat{n} \cdot \vec{x}') \vec{J} + (\hat{n} \cdot \vec{J}) \vec{x}' \right]}_A + \underbrace{\frac{1}{2} (\vec{x}' \times \vec{J}) \times \hat{n}}_B.\tag{III.36}$$

- We first consider the contribution of part B and identify the magnetization $\vec{\mathcal{M}}$,

$$\vec{\mathcal{M}} = \frac{1}{2} (\vec{x}' \times \vec{J}).\tag{III.37}$$

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- Then, for the B part,

$$\vec{A}(\vec{x}) = \frac{ik\mu_o}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right), \text{ with}\tag{III.38}$$

$$\vec{m} = \int \vec{\mathcal{M}} d^3x.\tag{III.39}$$

- In the radiation zone, $kr \gg 1$, we obtain:

$$\vec{A}(\vec{x}) = \frac{ik\mu_o}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (\text{III.40})$$

$$\vec{E}(\vec{x}) = -\frac{k^2}{4\pi} \sqrt{\frac{\mu_o}{\epsilon_o}} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (\text{III.41})$$

$$\vec{H}(\vec{x}) = -\sqrt{\frac{\epsilon_o}{\mu_o}} (\vec{E} \times \hat{n}). \quad (\text{III.42})$$

- This contribution is called magnetic dipole radiation.

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- For the A part in the contribution from $n = 1$, after standard handling we get:

$$A = \frac{i\omega}{2} \int \vec{x}' (\hat{n} \cdot \vec{x}') \rho(\vec{x}') d^3x', \quad (\text{III.43})$$

which represents order 2 moments of $\rho(\vec{x}')$, i.e. an electric quadrupole contribution, which we will not develop.

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III.2.4 Spectrum of continuous dipole radiation

- Consider a time-dependent electric dipole $\vec{p}(t)$, for simplicity with constant direction. We decompose $\vec{p} = \int e^{-i\omega t} \vec{p}(\omega) d\omega$.
- For a single harmonic component,

$$\vec{p}(t) = e^{-i\omega t} \vec{p}(\omega), \quad (\text{III.44})$$

the radiated $\vec{E}(\omega)$ field is (from Eq. III.32),

$$\vec{E}(\vec{x}, \omega) = \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{ck^2}{4\pi} \frac{e^{ikr}}{r} \hat{r} \times \vec{p}(\omega) \times \hat{r}, \quad (\text{III.45})$$

$$\vec{E}(\vec{x}, \omega) = \frac{k^2}{4\pi\epsilon_o} \frac{e^{ikr}}{r} \sin(\Theta) |p(\omega)| \hat{\Theta}, \quad (\text{III.46})$$

where Θ is co-latitude with $\hat{z} \parallel \vec{p}$.

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III.2.4.1 Continuous Spectra and Power Spectral Density

- The time-averaged flux radiated in direction $\hat{r}$ is $\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$ (Eq. 2.36).
- However, each Fourier component is in fact a Fourier density, with contributions $d\vec{p} = e^{-i\omega t} \vec{p}(\omega) d\omega$. Directly summing these contributions would result in $d\omega^2$ in the radiated power.
- For steady-state sources, another problem in the direct superposition of Fourier components is that the standard Fourier transform diverges:

$$\vec{p}(\omega) = \int_{-\infty}^{\infty} \vec{p}(t) e^{i\omega t} dt \rightarrow \infty. \quad (\text{III.47})$$

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- We can still introduce the concept of a spectrum in an averaged sense. We define the truncated Fourier transform:

$$\vec{p}_T(\omega) = \int_{-T/2}^{T/2} \vec{p}(t) e^{i\omega t} dt, \quad (\text{III.48})$$

and the time average of its amplitude:

$$\langle |p|^2 \rangle(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} |\vec{p}_T(\omega)|^2. \quad (\text{III.49})$$

- The differential power for a continuous source becomes:

$$\frac{dP}{d\omega d\Omega} = \frac{\omega^4}{32\pi^2 \epsilon_0 c^3} \langle |p|^2 \rangle(\omega) \sin^2 \Theta. \quad (\text{III.50})$$

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III.3 Radiation from a single charge

III.3.1 Liénard-Wiechert potentials

- Consider a particle with charge q and trajectory $\vec{r}(t)$, with velocity $\vec{u}(t)$.
- We can apply the retarded solution with source terms $\rho(\vec{x}, t) = q\delta(\vec{x} - \vec{r}(t))$, and $\vec{j}(\vec{x}, t) = \rho(\vec{x}, t)\vec{u}(t)$, and thus obtain the resulting potentials (tarea):

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (\text{III.51})$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \left[\frac{q\vec{u}}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (\text{III.52})$$

where $\vec{R}(t') = \vec{x} - \vec{r}(t')$, $R = |\vec{R}|$, $\hat{n}(t') = \frac{\vec{R}}{R}$, and $\vec{\beta}(t') = \frac{\vec{u}(t')}{c}$. These are the Liénard-Wiechert (L.-W.) potentials.

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- We can also calculate the corresponding electromagnetic field (tarea):

$$\vec{E}(\vec{x}, t) = \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{(1 - \beta^2)(\hat{n} - \vec{\beta})}{R^2(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{vel}}} + \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{\left(\hat{n} \times \left((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right)\right)}{cR(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{rad}}} \quad (\text{III.53})$$

with

$$\vec{B}(\vec{x}, t) = \frac{1}{c} \hat{n} \times \vec{E}(\vec{x}, t). \quad (\text{III.54})$$

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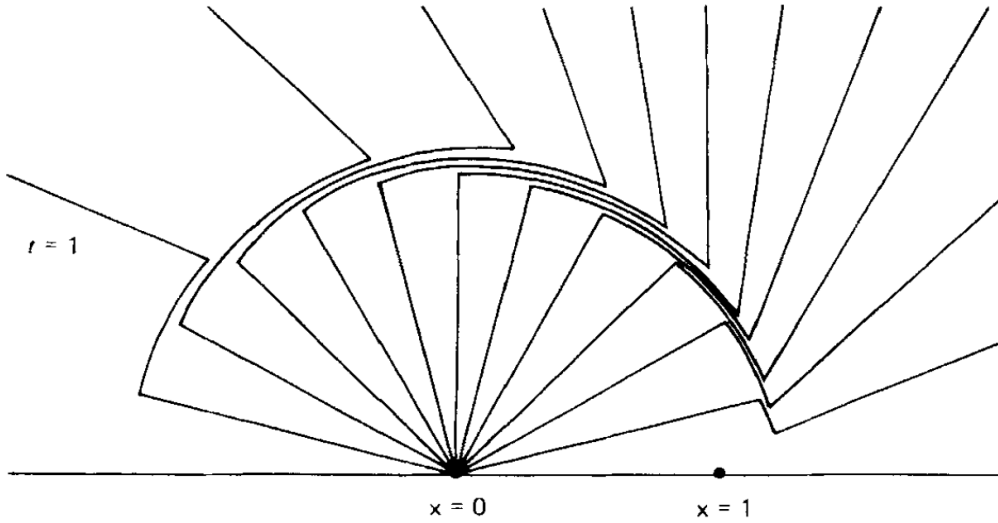
- We see that far away from the particle, the term labelled E_{rad} will eventually dominate. In fact, for a Fourier component, or for harmonic motion with $\vec{r}(t) \propto \exp(i\omega t)$, we find that (tarea)

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} = \beta \frac{R}{\lambda}. \quad (\text{III.55})$$

The radiation term dominates if $R \gg \lambda/\beta$, sometimes also called the *far zone*.

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- Illustration of the radiation field (from Rybicky & Lightman). A charge stops at $x = 0$ and $t = 0$. The figure shows the field lines at $t = 1$, where the charge would have been at $x = 1$ had it not stopped.



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III.3.2 Larmor Formula

- In the far zone, with $R \rightarrow \infty$, and in the Galilean limit ($\beta \ll 1$), the power radiated per unit solid angle is (tarea):

$$\frac{dP}{d\Omega} = \frac{\mu_o}{16\pi^2 c} q^2 a^2 \sin^2(\theta), \quad (\text{III.56})$$

in which $a = |\dot{\vec{u}}|$ and θ is the angle between $\hat{n}$ and $\vec{a}$.

- By applying Eq. III.56 to the case of a harmonically oscillating charge with dipole $\vec{p} = q\vec{r}_0 \exp(i\omega t)$, we can recover Eq. III.33 (tarea).
- We see that the *wave zone* as defined in Sec. III.2.1 matches the Galilean limit (see the discussion on the dipole approximation in Sec. 3.3 of Rybicki & Lightman).

- Integrating Eq. III.56 over all directions, we have the total power in SI:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}, \quad (\text{III.57})$$

which is the Larmor formula.

- In CGS,

$$P = \frac{2q^2 a^2}{3c^3}. \quad (\text{III.58})$$

III.3.3 Dipole of a charge assembly

- Consider an assembly of charges confined within a region of size d (here we follow Rybicki & Lightman 3.3).
- If τ is the minimum timescale over which particle velocities $\langle v \rangle$ change appreciably, we can ignore differences in retarded times if $c\tau \gg d$.
- If $l < d$ is the scale associated to a trajectory, then $\frac{\langle v \rangle}{c} \ll \frac{l}{d}$, or $\langle v \rangle \ll c$.
- Therefore, if $c\tau \gg d$ (equivalent to $\lambda \gg d$ for harmonically oscillating charges), we take the Galilean limit $\beta \rightarrow 0$ and the radiation field of a charge distribution is

$$\vec{E}_{\text{rad}} = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\hat{n} \times (\hat{n} \times \vec{\beta}_i)}{c|\vec{x} - \vec{x}_i|}. \quad (\text{III.59})$$

- To zeroth order,

$$\vec{E}_{\text{rad}}(x) = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{p}})}{4\pi\epsilon_0 c^2 r}. \quad (\text{III.60})$$

- The differential power is

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{p}}|^2}{(16\pi^2\epsilon_0 c^3)} \sin^2(\Theta), \quad (\text{III.61})$$

with $\hat{z} \parallel \vec{p}$ and where Θ is colatitude. In CGS,

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{p}}|^2}{4\pi c^3} \sin^2(\Theta), \quad (\text{III.62})$$

- The total radiated power is

$$P = \frac{1}{6\pi\epsilon_0 c^3} |\ddot{\vec{p}}|^2. \quad (\text{III.63})$$

and in CGS,

$$P = \frac{2}{3} \frac{|\ddot{\vec{p}}|^2}{c^3} \quad (\text{III.64})$$

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- In general, for a time-dependent dipole $\vec{p}(t)$ oscillating in a constant direction, Eq. III.60 gives

$$\vec{E}_{\text{rad}}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2 r} |\ddot{\vec{p}}(t)| \sin(\Theta), \quad (\text{III.65})$$

in SI, or, in CGS,

$$\vec{E}_{\text{rad}}(\vec{x}, t) = \frac{1}{c^2 r} |\ddot{\vec{p}}(t)| \sin(\Theta), \quad (\text{III.66})$$

with $p_{\text{SI}} = \sqrt{4\pi\epsilon_0} p_{\text{CGS}}$ and $E_{\text{SI}} = E_{\text{CGS}}/\sqrt{4\pi\epsilon_0}$.

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III.3.4 Radiation spectrum of a pulse

- Consider a “pulse” $\vec{E}(t)$, with finite time support. The spectrum,

$$\vec{E}(\omega) = \frac{1}{2\pi} \int \vec{E}(t) e^{i\omega t} dt, \quad (\text{III.67})$$

is Hermitian, $\vec{E}(-\omega) = \vec{E}^*(\omega)$, for $\vec{E}(t)$ to be real.

- The power radiated per unit area is given by the Poynting vector,

$$\frac{dW}{dt dA} = |\vec{S}(t)| = \frac{c}{4\pi} |\vec{E}(t)|^2 \text{ in CGS.} \quad (\text{III.68})$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} |\vec{E}(t)|^2 \text{ in SI.} \quad (\text{III.69})$$

- In CGS, the energy radiated by the pulse is

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} |\vec{E}(t)|^2 dt. \quad (\text{III.70})$$

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- From Parseval's theorem,

$$\int |\vec{E}(t)|^2 dt = 2\pi \int |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.71})$$

and with the Hermitian property,

$$\int |\vec{E}(t)|^2 dt = 4\pi \int_0^\infty |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.72})$$

- Therefore, in CGS

$$\frac{dW}{dA} = c \int_0^\infty |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.73})$$

and the spectrum of the energy radiated by the pulse is

$$\frac{dW}{dAd\omega} = c |\vec{E}(\omega)|^2. \quad (\text{III.74})$$

- For SI,

$$\frac{dW}{dAd\omega} = 4\pi \sqrt{\frac{\epsilon_0}{\mu_0}} |\vec{E}(\omega)|^2. \quad (\text{III.75})$$

- We can revisit the spectrum of dipole radiation with

$$\vec{p} = \int e^{-i\omega t} \vec{p}(\omega) d\omega. \quad (\text{III.76})$$

- Since $\ddot{\vec{p}}(t) = - \int \omega^2 \vec{p}(\omega) e^{-i\omega t} d\omega$, in CGS

$$\vec{E}(\omega) = -\frac{1}{c^2 r} \omega^2 \vec{p}(\omega) \sin(\Theta). \quad (\text{III.77})$$

- From Eq. III.74, the spectrum for the radiated energy is therefore, in CGS,

$$\frac{dW}{d\omega d\Omega} = \frac{1}{c^3} \omega^4 |\vec{p}(\omega)|^2 \sin^2 \Theta. \quad (\text{III.78})$$

- The total radiated energy, in CGS, is

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3} \omega^4 |\vec{p}(\omega)|^2. \quad (\text{III.79})$$

- In SI, the spectrum for the angular-differential radiated energy is

$$\frac{dW}{d\omega d\Omega} = \frac{1}{4\pi\epsilon_0 c^3} \omega^4 |\vec{p}(\omega)|^2 \sin^2 \Theta, \quad (\text{III.80})$$

which can be compared to the dipole approximation for continuous sources, which, from Eq. III.50, is

$$\frac{dP}{d\omega d\Omega} = \frac{1}{8\pi} \frac{1}{4\pi\epsilon_0 c^3} \omega^4 \langle |p|^2 \rangle(\omega) \sin^2(\Theta). \quad (\text{III.81})$$

in SI. The extra factor $1/(8\pi)$ in the power of continuous sources comes from the time-average $(1/2)$, and from the application of Parseval's theorem for the total pulse energy $(1/(4\pi))$. Also, Eq. III.80 refers to the radiated energy, with $\vec{p}(\omega)$ as a Fourier density in C m s (see Eq. III.76), whereas Eq. III.81 refers to the radiated power of a continuous source, with time-averaged squared amplitude $\langle |p|^2 \rangle(\omega)$ (Eq. III.49), in units of C² m² s.

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III.3.5 Radiation reaction

- This section is adapted to SI from Rybicki & Lightman Sec. 3.5 (see also Jackson Chap. 16 or Griffiths Sec. 11.2.2).
- Consider a periodic sistem, such that its mechanical state is identical between times t_1 and t_2 .
- Still in the Galilean limit, the total energy radiated by the charge between t_1 and t_2 is (in SI),

$$W = \frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2 dt. \quad (\text{III.82})$$

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- Energy conservation requires the existence of a 'radiation reaction' force F_{rad} , such that W is extracted from the particle's kinetic energy,

$$\int_{t_1}^{t_2} dt \vec{F}_{\text{rad}} \cdot \vec{u} = -W. \quad (\text{III.83})$$

- One expression for the radiation reaction is the Abraham-Lorentz formula,

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}. \quad (\text{III.84})$$

We can confirm that Eq. III.84 fulfills Eq. III.83 (tarea).

- In CGS,

$$\vec{F}_{\text{rad}} = \frac{2q^2}{3c^3} \dot{\vec{a}} = m \tau \dot{\vec{a}}, \quad (\text{III.85})$$

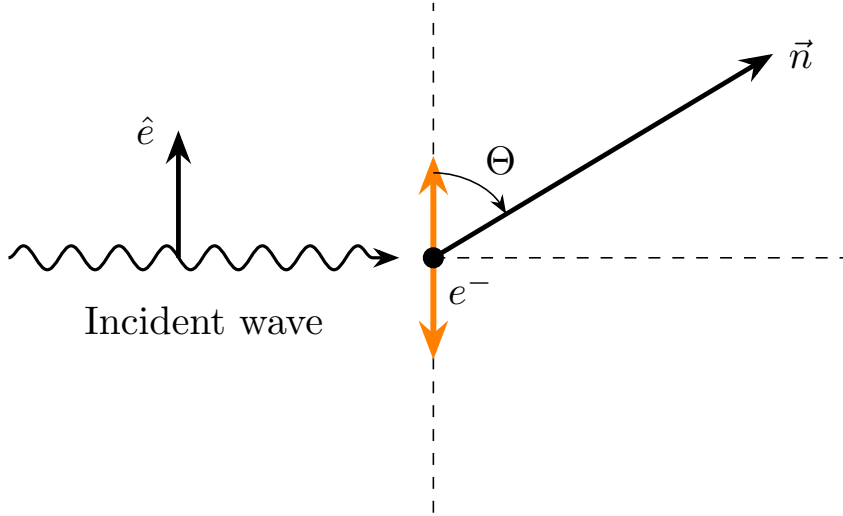
where $\tau = \frac{2}{3} r_0 / c$.

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III.4 Non-relativistic applications

III.4.1 Thomson scattering

This section follows from Rybicki & Lightman Sec. 3.4.



- A linearly polarized wave exerts the force $\vec{F} = q\hat{e}E_o \sin(\omega t)$ on a charge q .

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- The equation of motion

$$m\ddot{\vec{r}} = \hat{e}E_o \sin(\omega t), \quad (\text{III.86})$$

yields the dipole $\vec{d} = \hat{e}\vec{r}$ from integration of the acceleration

$$\vec{d} = q^2 \hat{e} \frac{E_o}{m} \sin(\omega t), \quad (\text{III.87})$$

and

$$\vec{d} = - \left(\frac{q^2 E_o}{m\omega_o^2} \right) \hat{e} \sin(\omega t). \quad (\text{III.88})$$

.38

- The amplitude of the oscilating dipole is $d = \frac{q^2 E_o}{m\omega^2}$.
- Dipolar radiation (Eq. III.33) yields, in SI (or MKS),

$$\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_o}{\epsilon_o}} k^4 |\vec{d}|^2 \sin^2(\Theta). \quad (\text{III.89})$$

- In CGS, the same power is (Eq. 3.23 in Rybicki & Lightman, with an additional 1/2 for time-averaging):

$$\frac{dP}{d\Omega} = \frac{1}{c} \frac{k^4 d^2}{8\pi} \sin^2(\Theta). \quad (\text{III.90})$$

The conversion can be obtained by substitution of $\vec{d}_{\text{CGS}} = \vec{d}_{\text{SI}}/\sqrt{4\pi\epsilon_o}$ and $c = 1/\sqrt{\epsilon_o\mu_o}$.

.39

- The incident power is

$$\langle \vec{S} \rangle = \frac{c}{8\pi} E_o^2 \hat{n}, \quad (\text{III.91})$$

in CGS, and

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathcal{E}|^2 \hat{n}, \quad (\text{III.92})$$

in SI (Eq. 2.36).

- We introduce the differential cross section

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega}. \quad (\text{III.93})$$

- The differential cross section for Thomson scattering is

$$\frac{d\sigma}{d\Omega} = r_o^2 \sin^2(\Theta), \quad (\text{III.94})$$

where

$$r_o = \frac{e^2}{m c^2} \quad (\text{III.95})$$

is the classical radius of the electron ($r_o \sim 3 \times 10^{-13}$ cm) in CGS, and

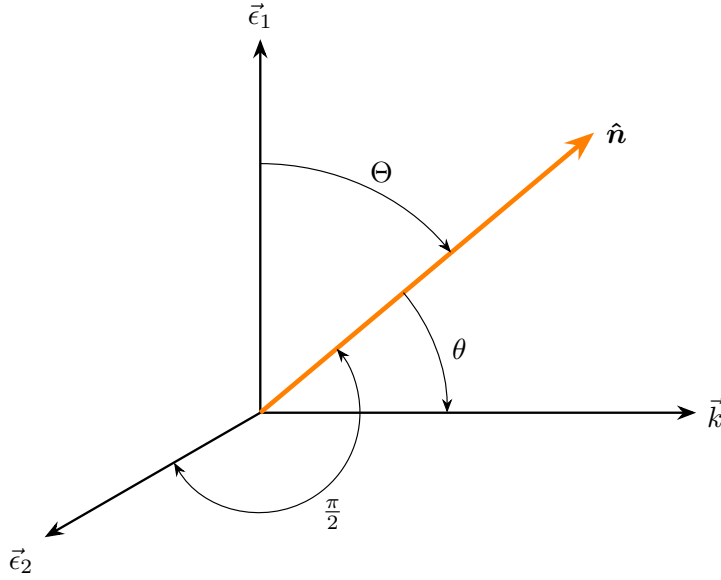
$$r_o = \frac{e^2}{4\pi\epsilon_0 m c^2} \quad (\text{III.96})$$

in SI.

- The total cross section is

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} r_o^2. \quad (\text{III.97})$$

- We can extend the previous derivation to the case of unpolarized radiation.
- An unpolarized beam is the sum of two independent linearly polarized components, along $\hat{e}_1$ and $\hat{e}_2$.
- We place $\hat{e}_1$ in the plane of scattering (see Fig.).



.42

- For unpolarized incident radiation, the Poynting flux of each of two components of the incident field, E_1 and E_2 , are equal.
- The scattered field is the sum of the scattered fields of each component,

$$\vec{E}_{\text{scat}}(t) = \vec{E}_{1,\text{scat}}(t) + \vec{E}_{2,\text{scat}}(t), \quad (\text{III.98})$$

and

$$\begin{aligned} \langle |\vec{E}_{\text{scat}}(t)|^2 \rangle &= \langle |\vec{E}_{1,\text{scat}}(t)|^2 \rangle + \langle |\vec{E}_{2,\text{scat}}(t)|^2 \rangle + \\ &\quad 2 \langle \vec{E}_{1,\text{scat}}(t) \cdot \vec{E}_{2,\text{scat}}(t) \rangle, \end{aligned} \quad (\text{III.99})$$

where the cross-term is null.

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- Therefore,

$$\langle dP \rangle = \langle dP_1 \rangle + \langle dP_2 \rangle, \quad (\text{III.100})$$

and, in terms of the total incident Poynting field, the total cross section is the average of the cross-sections for each linear polarization components,

$$\left. \frac{d\sigma}{d\Omega}(\Theta) \right|_{\text{unpol}} = \frac{1}{2} \left[\left. \frac{d\sigma}{d\Omega}(\Theta) \right|_{\text{pol}} + \left. \frac{d\sigma}{d\Omega} \left(\frac{\pi}{2} \right) \right|_{\text{pol}} \right], \quad (\text{III.101})$$

$$= \frac{1}{2} r_0^2 (1 + \sin^2 \Theta), \quad (\text{III.102})$$

$$= \frac{1}{2} r_0^2 (1 + \cos^2 \theta). \quad (\text{III.103})$$

$$(\text{III.104})$$

.44

- The total scattered intensity (i.e. total power) is proportional to $1 + \cos^2 \theta$. From this intensity the weaker component (in the plane of scattering) has intensity proportional to $\cos^2 \theta$. The unpolarized scattered intensity must have equal components, and is proportional to $2 \cos^2 \theta$. The polarized scattered intensity is thus proportional to $1 + \cos^2 \theta - 2 \cos^2 \theta$.
- The polarization fraction is

$$\Pi = \frac{1 - \cos^2(\theta)}{1 + \cos^2(\theta)}. \quad (\text{III.105})$$

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III.4.2 Harmonically bound particles

- This section follows from Rybicki & Lightman Sec. 3.6, and uses CGS units.
- The equation of motion for a radiating harmonic oscillator is

$$-\tau \ddot{x} + \ddot{x} + \omega_0 x = 0, \quad (\text{III.106})$$

where τ in the radiation reaction force is given by Eq. III.85.

- We assume that the damping due to radiation will be small, and use $x(t) \propto \cos(\omega_0 t + \phi_0)$, so $\ddot{x} \approx -\omega_0^2 x$.
- The equation of motion becomes

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0. \quad (\text{III.107})$$

.46

- Searching for solutions to Eq. III.107 in the form $e^{\alpha t}$, we require

$$\alpha \approx \pm i\omega_0 - \frac{1}{2}\omega_0^2 \tau, \quad (\text{III.108})$$

if $\omega_0 \tau \ll 1$.

- With the initial condition $x(0) = x_0$ and $\dot{x}(0) = 0$,

$$x(t) = x_0 e^{-\frac{\Gamma}{2}t} \cos(\omega_0 t), \quad (\text{III.109})$$

$$= \frac{1}{2}x_0 \left(e^{-\frac{1}{2}\Gamma t + i\omega_0 t} + e^{-\frac{1}{2}\Gamma t - i\omega_0 t} \right), \quad (\text{III.110})$$

where $\Gamma = \omega_0^2 \tau$.

.47

- The Fourier transform of $x(t)$ is

$$x(\omega) = \frac{1}{2\pi} \int x(t) e^{i\omega t} dt = \frac{x_0}{4\pi} \left[\frac{1}{\Gamma/2 - i(\omega + \omega_0)} + \frac{1}{\Gamma/2 - i(\omega - \omega_0)} \right]. \quad (\text{III.111})$$

- Restricting to positive frequencies, and neglecting the tail from the negative singularity,

$$x(\omega) \approx \frac{x_0}{4\pi} \frac{1}{\Gamma/2 - i(\omega - \omega_0)}, \quad (\text{III.112})$$

whose squared modulus is

$$|x(\omega)|^2 \approx \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0)^2}. \quad (\text{III.113})$$

- From Eq. III.79, the radiated energy spectrum is then

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0)^2}. \quad (\text{III.114})$$

- With $k \equiv m\omega_0^2$ (spring constant) and $\Gamma = \omega_0^2\tau$,

$$\frac{dW}{d\omega} = \left(\frac{1}{2} k x_0^2 \right) \frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2} = \epsilon_k \phi(\omega), \quad (\text{III.115})$$

where ϵ_k is the potential energy of the spring, and $\phi(\omega)$ is the profile of the spectrum, with $\int \phi(\omega) d\omega = 1$.

- The profile $\phi(\omega)$ is the Lorentz profile.

- Now consider a driven harmonically bound particle, such as a charge under the influence of a linearly polarized plane wave,

$$m\ddot{x} = -m\omega_0^2 x + m\tau\ddot{x} + eE_0 \cos(\omega t). \quad (\text{III.116})$$

- The steady solution is $x = x_0 e^{i\omega t} \equiv |x_0| e^{i(\omega t + \delta)}$, with

$$x_0 = -\frac{e E_0}{m} / (\omega^2 - \omega_0^2 - i\omega_0^3 \tau), \quad (\text{III.117})$$

$$\delta = \arctan \left(\frac{\omega^2 \tau}{\omega^2 - \omega_0^2} \right). \quad (\text{III.118})$$

- In steady state, the charge acts as an oscillating dipole, radiating a total power given by the Larmor formula (Eq. III.58 or Eq. III.64):

$$P = \frac{e^2 |x_0|^2 \omega^4}{3c^3} = \frac{e^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3 \tau)^2}. \quad (\text{III.119})$$

- Comparing the radiate power from Eq. III.119 with the incident time-average flux, $\langle S \rangle = \frac{c}{8\pi} E_0^2$, we introduce the scattering cross section

$$\sigma(\omega) = \frac{P(\omega)}{\langle S \rangle} = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3 \tau)^2}. \quad (\text{III.120})$$

- In the limit $\omega \gg \omega_0$, we recover Thomson scattering.
- In the limit $\omega \ll \omega_0$, we enter the dipolar approximation regime and

$$\sigma(\omega) = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4. \quad (\text{III.121})$$

This regime is known as Rayleigh Scattering.

.51

- In the regime $\omega \sim \omega_0$ (tarea),

$$\sigma(\omega) \approx \frac{2\pi^2 e^2}{m c} \frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2}, \quad (\text{III.122})$$

where we recognize the Lorentz profile.

- If we integrate this approximation over all frequencies $\nu = \omega/(2\pi)$,

$$\int_0^\infty d\nu \sigma(\nu) = \frac{\pi e^2}{m c}. \quad (\text{III.123})$$

- In the quantum mechanical calculation of radiative cross sections between two state n_1 and n_2 , the resulting cross sections are sometimes referred to the classical treatment of Eq. III.123 through the oscillator strength $f_{n_1 n_2}$:

$$\int_0^\infty d\nu \sigma(\nu) = f_{n_1 n_2} \frac{\pi e^2}{m c}. \quad (\text{III.124})$$

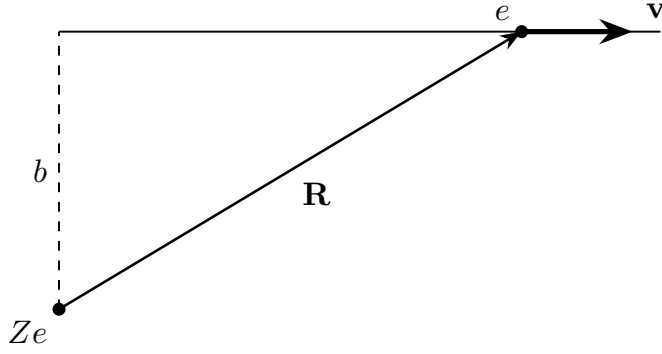
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III.4.3 Thermal free-free

- This section follows from Rybicki & Lightman Secs. 5.1-5.3.
- In an ionized plasma, radiation stems from electron-proton encounters, with dipole $\vec{d} = -e\vec{R}$, with acceleration $\ddot{\vec{d}} = -e\ddot{\vec{v}}$.
- In the Fourier plane,

$$-\omega^2 \vec{d}(\omega) = -\frac{e}{2\pi} \int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t} dt. \quad (\text{III.125})$$

- We introduce the impact parameter b .



.53

- A timescale for the collision is the collision time $\tau \equiv \frac{b}{v}$.
- If $\omega\tau \gg 1$, the integrand in Eq. III.125 oscillates rapidly and averages to 0, while if $\omega\tau \ll 1$ this integrand is essentially unity:

$$\vec{d}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2} \Delta\vec{v}, & \text{if } \omega\tau \ll 1, \\ 0, & \text{if } \omega\tau \gg 1. \end{cases} \quad (\text{III.126})$$

- We can calculate the spectrum of the emitted radiation using Eq. III.79,

$$\frac{dW}{d\omega} = \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta\vec{v}|^2, & \omega\tau \ll 1, \\ 0, & \omega\tau \gg 1. \end{cases} \quad (\text{III.127})$$

.54

- We estimate $\Delta\vec{v}$, the velocity change in a collision, by assuming that the electron follows a straight line. The longitudinal acceleration cancels out on both sides of the collision. However, the perpendicular component pulls the electron all along its trajectory. We integrate this normal acceleration along a straight path (and neglecting the concomitant deflection of the path):

$$\Delta v = \frac{Ze^2}{m} \int_{-\infty}^{\infty} \frac{b dt}{(b^2 + v^2 t^2)^{3/2}} = \frac{2Ze^2}{mbv}. \quad (\text{III.128})$$

- The radiated energy is then

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2 e^6}{3\pi c^3 m^2 v^2 b^2}, & b \ll v/\omega \\ 0, & b \gg v/\omega. \end{cases} \quad (\text{III.129})$$

.55

- For a fixed electron speed v , we can sum over b to get the total emissivity:

$$\frac{dW}{d\omega dV dt} = n_e n_i 2\pi v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} b db, \quad (\text{III.130})$$

starting from a minimum impact parameter $b_{\min}$.

- Since the integrand in Eq. III.130 is zero above a maximum impact parameter $b_{\max} = \frac{v}{\omega}$,

$$\begin{aligned} \frac{dW}{d\omega dV dt} &= \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\ &= \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \ln \left(\frac{b_{\max}}{b_{\min}} \right) \end{aligned} \quad (\text{III.131})$$

.56

- The value of $b_{\min}$ can be estimated as the smallest value for the straight-line trajectory approximation, i.e. $\Delta v \sim v$ and $b_{\min} = 4Ze^2/(\pi m v^2)$.
- The floor value of $b_{\min}$ is set by the uncertainty principle, with $\Delta x \Delta p \gtrsim \hbar$, $\Delta x \sim b$ and $\Delta p \sim mv$, or $b_{\min} = \hbar/(mv)$.
- An adequate treatment of the total radiated power requires field quantization, and is beyond the scope of this course. The connection with the classical treatment presented here is implemented through a Gaunt factor $g_{ff}(v, \omega)$, typically of order unity, and

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m^2 v} n_e n_i Z^2 g_{ff}(v, \omega), \quad (\text{III.132})$$

with

$$g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln \left(\frac{b_{\max}}{b_{\min}} \right). \quad (\text{III.133})$$

.57

- For a thermal distribution of speeds, we average Eq. III.132 over the Maxwellian distribution, with probability density

$$dP \propto v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv. \quad (\text{III.134})$$

- For radiation at frequency ν , we require $h\nu < \frac{1}{2}mv^2$ for photon creation,

$$\frac{dW(T, \omega)}{dV dt d\omega} = \frac{\int_{v_{\min}}^{\infty} \frac{dW(v, \omega)}{d\omega dV dt} v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv}{\int_0^{\infty} v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv}, \quad (\text{III.135})$$

and we obtain the emission coefficient (tarea):

$$\begin{aligned} j_{\nu}^{ff}(\nu) &= \frac{1}{4\pi} \frac{dW}{dV dt d\nu} = \\ &= \frac{2^5 \pi e^6}{3mc^3} \left(\frac{2\pi}{3km} \right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}. \end{aligned} \quad (\text{III.136})$$

.58

- The associated absorption coefficient is give by Kirchof's law:

$$j_{\nu}^{ff} = \alpha_{\nu}^{ff} B_{\nu}(T). \quad (\text{III.137})$$