# Estimating the Power Spectrum of an Image

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## 1. Characteristics of the image

Consider an image of $N_1 \times N_2$ pixels, $\Delta(n_1, n_2)$, $-N_1/2 \le n_1 \le N_1/2 - 1$, $-N_2/2 \le n_2 \le N_2/2 - 1$. Usually both $N_1$ and $N_2$ will be powers of 2. If not, the image can be padded with zeros up to the next power of two. The effect of this is discussed below.

In CMB work, it is usual to work with $\Delta T/T_0$. I assume that the pixel values in the image represent sky brightness ($\Delta T$) in units of K (if this is not the case, the units should be converted to K by applying a scale factor) and normalize by $T_0 = 2.726$ K to get the conventional dimensionless quantity.

To estimate the two-dimensional power spectrum, we form the discrete Fourier transform of the image using an FFT algorithm and square it. Each pixel of the resulting array represents an estimate of the contribution to the variance of the image from a particular spatial frequency vector. We are interested in the *expectation value* of the variance as a function of the magnitude of the spatial frequency. If we assume that the underlying statistical process is isotropic, then the direction of the spatial frequency vector doesn't matter, and we can estimate the power spectrum by forming azimuthal averages at constant spatial-frequency magnitude.

## 2. Use of discrete Fourier transform

The discrete Fourier transform (which is what an FFT algorithm calculates) of the image, $V = \mathrm{FFT}^{\pm}(\Delta)$, is defined by

$$V(k_1, k_2) = \sum_{n_2=-N_2/2}^{N_2/2-1} \sum_{n_1=-N_1/2}^{N_1/2-1} \Delta(n_1, n_2) \exp\left[\frac{\pm 2\pi i k_2 n_2}{N_2}\right] \exp\left[\frac{\pm 2\pi i k_1 n_1}{N_1}\right], \tag{1}$$

for $-N_1/2 \le k_1 \le N_1/2 - 1$, $-N_2/2 \le k_2 \le N_2/2 - 1$. (The $\pm$ sign represents the sign of the complex exponential; for our present purposes, we can use either sign.)

Note the following points (left as exercises for the reader!):

**1.** The image $\Delta$ is real, so $V$ is conjugate-symmetric:

$$V(-k_1, -k_2) = V^*(k_1, k_2). \tag{2}$$

**2.** The transform is periodic:

$$V(k_1 + j_1 N_1, k_2 + j_2 N_2) = V(k_1, k_2) \tag{3}$$

for any integers $(j_1, j_2)$.

**3.** The zero-frequency term in the transform is the sum of the image pixels:

$$V(0,0) = \sum_{n_2=-N_2/2}^{N_2/2-1} \sum_{n_1=-N_1/2}^{N_1/2-1} \Delta(n_1, n_2). \tag{4}$$

**4.** Parseval's or Rayleigh's theorem:

$$\sum_{k_1} \sum_{k_2} |V(k_1, k_2)|^2 = N_1 N_2 \sum_{n_1} \sum_{n_2} |\Delta(n_1, n_2)|^2. \tag{5}$$

Note the factor of $N_1 N_2$.

**5.** The inverse transform is $\mathrm{FFT}^{\mp}(V)$:

$$\mathrm{FFT}^{+}(\mathrm{FFT}^{-}(\Delta)) = N_1 N_2 \Delta. \tag{6}$$

## 3. Sampling in the *uv* plane

If each pixel in the image has size $\delta x_1 \times \delta x_2$ (radians), then the sampling in the transform ($uv$) plane is

$$\delta u_1 = \frac{1}{N_1 \delta x_1}, \quad \delta u_2 = \frac{1}{N_2 \delta x_2}, \tag{7}$$

in spatial-frequency units (cycles per radian on the sky). The sampling in *angular* spatial frequency is

$$\delta l_1 = \frac{2\pi}{N_1 \delta x_1}, \quad \delta l_2 = \frac{2\pi}{N_2 \delta x_2}. \tag{8}$$

For an equal-area projection of small part of the sky, there is a good correspondence between these angular spatial frequencies and multipole number $l$,

$$l(l+1) \approx (k_1 \delta l_1)^2 + (k_2 \delta l_2)^2. \tag{9}$$

For example, for a square image with $512 \times 512$ square pixels of size $\delta x = 1$ arcmin, $\delta u = 6.71$ wavelengths and $\delta l = 42.2$. The FFT of the image will sample $l$ up to $l_{\mathrm{max}} = \pi/\delta x = 256 \delta l = 10,800$. Note that $l_{\mathrm{max}}$ depends only on the size of the pixels $\delta x$, not the number of pixels.

## 4. Two-dimensional power spectrum

It is conventional to normalize the power spectrum so that its integral is the total variance of the image, $\sigma^2$. Assuming that the image has zero mean,

$$\sigma^2 = \frac{1}{N_1 N_2} \sum_{n_1} \sum_{n_2} |\Delta(n_1, n_2)|^2. \tag{10}$$

Applying Parseval's theorem (equation 5), we can see that

$$\sigma^2 = \frac{1}{N_1^2 N_2^2} \sum_{k_1} \sum_{k_2} |V(k_1, k_2)|^2. \tag{11}$$

which shows that the variance per pixel in the $(u, v)$ plane is $|V(k_1, k_2)|^2 / N_1^2 N_2^2$.

## 5. One-dimensional power spectrum

We are interested in the contribution to the variance from (angular) spatial frequencies between $l_1$ and $l_2$, say; this is given by the sum

$$\sigma^2(l_1, l_2) = \frac{1}{N_1^2 N_2^2} \sum_{k_1} \sum_{k_2} |V(k_1, k_2)|^2, \tag{12}$$

where the sum is taken over all integers $k_1$ and $k_2$ such that

$$l_1 + \tfrac{1}{2} \le \sqrt{(k_1 \delta l_1)^2 + (k_2 \delta l_2)^2} < l_2 + \tfrac{1}{2}.$$

(For the small angular scales and large $l$s that we are interested in, the odd $\frac{1}{2}$ can be safely ignored.)

## 6. Relationship to angular power spectrum

In the spherical harmonic representation, the image variance is given by

$$\sigma^2 = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l, \tag{13}$$

where $C_l$ is the angular power spectrum. We have formed an estimate of $C_l$ averaged over the band $l_1 \le l < l_2$,

$$\sigma^2(l_1, l_2) \approx \sum_{l=l_1}^{l_2} \frac{2l+1}{4\pi} C_l. \tag{14}$$

Thus the estimate of $C_l$ is

$$C_l \approx \frac{4\pi\sigma^2(l_1, l_2)}{(2l+1)\Delta l}, \tag{15}$$

where $\Delta l = l_2 - l_1$ is the band width in $l$ and $l = (l_1 + l_2)/2$ is the band center.

The quantity usually plotted (confusingly called $\mathcal{C}_l$ by Bond) is

$$\mathcal{C}_l \equiv l(l+1)C_l/2\pi \approx \frac{(l+\frac{1}{2})\sigma^2(l_1, l_2)}{\Delta l}. \tag{16}$$

Thus to estimate the power spectrum, we choose $l_1$ and $l_2$ (or $\Delta l$) and use equations 12 and 16. [The effective $l$ of the estimate is not simply the arithmetic mean of the band, but this probably doesn't matter at large $l$.]

## 7. Convolution with a gaussian

We wish to convolve the image $\Delta(x_1, x_2)$ ($x_1 = n_1\delta x_1$, $x_2 = n_1\delta x_2$), with a circular gaussian beam $f(x_1, x_2)$ of FWHM $a = \sigma\sqrt{8\ln 2}$ radians,

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right), \\ &= \frac{4\ln 2}{\pi a^2}\exp\left(\frac{-4\ln 2(x_1^2 + x_2^2)}{a^2}\right), \end{aligned} \tag{17}$$

where $F$ is normalized to unit integral to preserve the total flux density.

This can be accomplished by multiplying the FT of the image by the FT of $f$,

$$F(u_1, u_2) = \exp\left(\frac{-(\pi a)^2(u_1^2 + u_2^2)}{4\ln 2}\right), \tag{18}$$

where $u_1 = k_1\delta u_1$, $u_2 = k_1\delta u_2$.

i.e.

$$\Delta^{\mathrm{conv}} = \mathrm{FFT}^{+}(F \cdot \mathrm{FFT}^{-}(\Delta))/N_1N_2. \tag{19}$$

In terms of angular spatial frequency $l$ (using equation 9,

$$F(l) = \exp\left(\frac{-a^2 l(l+1)}{16\ln 2}\right). \tag{20}$$

The effect on the power spectrum is to multiply $C_l$ by

$$W_l = F^2(l) = \exp\left(\frac{-a^2 l(l+1)}{8\ln 2}\right) = \exp\left(-\sigma^2 l(l+1)\right), \tag{21}$$

where $\sigma$ is the standard deviation of the gaussian beam. This function falls to 0.5 at $l \approx 2\sqrt{2}\ln 2/a \approx 6740/a$ if $a$ is measured in arcmin.

## 8. Image masking

Usually we will not have a complete image with power-of-two dimensions. In this case, we proceed as before but padding the image with zeros. Thus we work with the array

$$\Delta'(n_1, n_2) = M(n_1, n_2)\Delta(n_1, n_2), \tag{22}$$

where the $M$ values (the "mask") can be 0 or 1, or more generally any value. The effect on the Fourier transform is to convolve $V(k_1, k_2)$ with the Fourier transform of $M$, so the choice of mask is a compromise between resolution and sidelobes in the $(u, v)$ plane. It is probably advisable to choose a mask function that tapers smoothly to zero at the edge of the image.

This alters the normalization of the derived power spectrum. Applying Parseval's theorem, as in equation 11, we have

$$\begin{aligned} \sum_{k_1}\sum_{k_2} |V'(k_1, k_2)|^2 &= N_1N_2 \sum_{n_1}\sum_{n_2} |\Delta'(n_1, n_2)|^2 \\ &= N_1N_2 \sum_{n_1}\sum_{n_2} |M(n_1, n_2)|^2 |\Delta(n_1, n_2)|^2 \end{aligned} \tag{23}$$

Taking the expectation value, and noting that $\langle |\Delta(n_1, n_2)|^2 \rangle = \sigma^2$,

$$\left\langle \sum_{k_1} \sum_{k_2} |V'(k_1, k_2)|^2 \right\rangle = N_1 N_2 \sigma^2 \sum_{n_1} \sum_{n_2} |M(n_1, n_2)|^2 \tag{24}$$

and hence

$$\sigma^2 = \frac{\sum_{k_1} \sum_{k_2} \langle |V'(k_1, k_2)|^2 \rangle}{N_1 N_2 \sum_{n_1} \sum_{n_2} |M(n_1, n_2)|^2}. \tag{25}$$

This is the generalization of equation 11 for use when the image is masked. The variance per pixel in the $(u, v)$ plane, and hence the power spectrum $C_l$, is weighted by the sum of the squares of the mask pixels instead of the total number of pixels as in the unmasked case. Equation 12 must be modified in the same manner.