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Part I

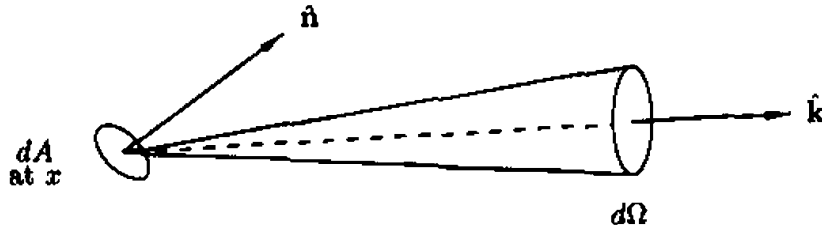
Radiative Transfer

I.1 Specific Intensity

I.1.1 Definition of specific intensity

- In geometrical optics we approximate the transport of radiative energy along straight lines, or ‘rays’, in direction $\hat{k}$.
- If dE is the amount of radiative energy crossing an area $d\vec{A} = dA\hat{n}$ in the direction $\hat{k}$, within a solid angle $d\Omega$, and with a frequency range within $[\nu, \nu + d\nu]$, then the monochromatic specific intensity $I_\nu(\hat{k}, \vec{x}, t)$ is defined by

$$dE = I_\nu(\hat{k}, \vec{x}, t) \hat{k} \cdot \hat{n} dA d\Omega d\nu dt. \quad (\text{I.1})$$



.2

- Geometrical optics can only be applied to cases where the wavelength λ is much smaller than the scale of structure in the propagation medium, ' a '. In other words, it cannot be used to describe the interaction of optical light with atoms, and neither can it be used to describe diffraction phenomena which become evident when $\lambda \sim a$.
- To see this we follow Rybicki & Lightman (Sec. 2.6). Because of the uncertainty principle, for a ray with $\hat{k} = \hat{z}$,

$$dx dp_x dy dp_y = p^2 dA d\Omega \gtrsim h^2, \quad (\text{I.2})$$

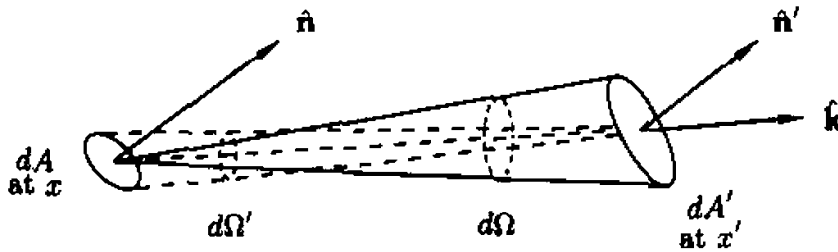
and

$$dA d\Omega \gtrsim \lambda^2. \quad (\text{I.3})$$

- We see that if $dA \sim \lambda^2$, then $d\Omega \sim 1$, and there is no constraint on direction. Similar constraints apply to time, with $d\nu dt \gtrsim 1$.
- Thus the transfer theory described below is applicable when $\lambda \ll a$.

.3

- In the absence of interaction with matter, $I(\hat{k}, \vec{x})$ is constant for any point $\vec{x}$ along the ray $\hat{k}$. Proof: see Shu Vol. I.



.4

I.1.2 Moments of specific intensity

- The mean intensity at $\vec{x}$ is simply

$$J_\nu(\vec{x}) = \frac{1}{4\pi} \int d\Omega I_\nu(\hat{k}, \vec{x}). \quad (\text{I.4})$$

In spherical coordinates, with $\mu = \cos(\theta)$,

$$J_\nu = \frac{1}{4\pi} \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^1 d\mu I_\nu(\mu, \Phi) \quad (\text{I.5})$$

- The monochromatic radiative energy density is

$$u_\nu = \frac{4\pi}{c} J_\nu. \quad (\text{I.6})$$

and the bolometric radiative energy density is

$$u = \int d\nu u_\nu. \quad (\text{I.7})$$

.5

- The net flux in the direction $\hat{n}$ and at point $\vec{x}$ is

$$F_\nu(\hat{n}) = \int d\Omega I_\nu(\hat{k}, \vec{x}) \cos(\theta), \quad (\text{I.8})$$

where θ is the angle between $\hat{k}$ and $\hat{n}$. With $\mu = \cos(\theta)$,

$$F_\nu(\hat{n}) = \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^1 d\mu \mu I_\nu(\mu, \Phi). \quad (\text{I.9})$$

- Given that the photon momentum is ϵ/c , with $\epsilon = h\nu$, the radiation pressure in direction $\hat{n}$ is

$$p(\hat{n}) = \frac{2}{c} \int d\Omega \cos^2(\theta) I_\nu(\hat{k}, \vec{x}), \quad (\text{I.10})$$

or, in terms of $\mu = \cos(\theta)$,

$$p(\hat{n}) = \frac{2}{c} \int_{\Phi=0}^{2\pi} d\Phi \int_{\mu=-1}^1 d\mu \mu^2 I_\nu(\mu, \Phi). \quad (\text{I.11})$$

.6

- Problem: consider a sphere with radius R which emits a uniform specific intensity field from its surface, B . Prove that the flux in direction $\hat{r}$ a distance r away from the center of the sphere is

$$F = \pi B \left(\frac{R}{r} \right)^2. \quad (\text{I.12})$$

The flux at the surface of the sphere is thus $F = \pi B$.

.7

I.2 Transfer equation

- In the presence of matter the spec. int. field may change with position (and time),

$$\frac{\partial I_\nu}{c \partial t} + \hat{k} \cdot \vec{\nabla} I_\nu = \text{sources} - \text{sinks}. \quad (\text{I.13})$$

- The sources correspond to the emission of radiative energy by matter, and to reflection in the direction $\hat{k}$.
- The sinks correspond to absorption of radiative energy by matter, or by reflection out of the direction $\hat{j}$.
- Reflected light is also called ‘scattered’ light.

.8

I.2.1 Emission

- In the presence of matter the specific intensity field may change with position. A volume element $dV = ds dA$ emits an energy

$$dE = j_\nu d\nu dV dt d\Omega, \quad (\text{I.14})$$

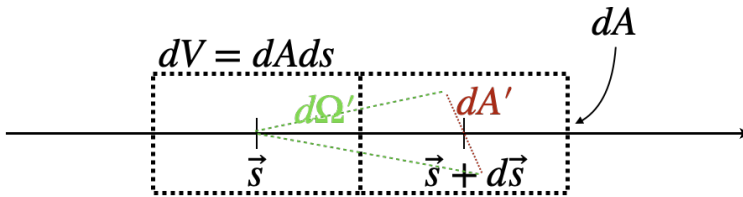
towards the direction of $d\Omega$, where $j_\nu(\hat{k})$ is the emission coefficient.

- From the definition of specific intensity,

$$dE = dI_\nu(\hat{k}', \vec{s}, t) \hat{k}' \cdot \hat{n}' dA' d\Omega' d\nu dt, \quad (\text{I.15})$$

for any surface $d\vec{A}'$ centered on $\vec{s}$. If we choose $d\vec{A} = d\vec{A}'$, $\hat{k} = \hat{s}$ and $d\Omega = d\Omega'$ then $\hat{k} \cdot \hat{n} = 1$. In the absence of absorption I_ν is constant along $\hat{s}$, so the increase in I_ν due to the contribution from dV at $\vec{s}$ is

$$dI_\nu = j_\nu ds. \quad (\text{I.16})$$



.9

I.2.2 Absorption

- For pure absorption, without scattering or emission, and if the absorption is caused by infinitely small particles that do not shadow each other inside a volume $dV = ds dA$, we expect

$$dI_\nu = -\alpha_\nu I_\nu ds, \quad (\text{I.17})$$

- The absorption coefficient α_ν can be connected to the effective cross-sectional area σ_ν of the absorbers. The total energy absorbed in volume dV is

$$dE = -I_\nu n \sigma_\nu dV d\Omega dt ds, \quad (\text{I.18})$$

where the minus sign accounts for dE being removed from the radiation field.

- The difference in specific intensity is given by

$$dE = dI_\nu(\hat{k}, \vec{x}, t) dA d\Omega d\nu dt, \quad (\text{I.19})$$

so

$$\alpha_\nu = n \sigma_\nu. \quad (\text{I.20})$$

.10

I.2.3 Scattering

- Reflection of photons, rather than absorption, will also decrease the specific intensity along a ray $\hat{k}$. In analogy with pure absorption, the corresponding decrease in specific intensity will be

$$dI_\nu = -\alpha_\nu^{\text{sca}} I_\nu ds, \quad (\text{I.21})$$

with $\hat{s} = \hat{k}$.

- Photons coming from all directions may also be reflected into the ray $\hat{k}$. To describe the probability of reflection from direction $\hat{k}'$ into direction $\hat{k}$ we use the phase function $\Phi_\nu(\hat{k}, \hat{k}')$ (also called the ‘scattering probability density’),

$$dI_\nu = \alpha_\nu^{\text{sca}} \oint \Phi_\nu(\hat{k}, \hat{k}') I_\nu(\hat{k}') d\Omega' ds. \quad (\text{I.22})$$

The phase function is normalized and symmetric:

$$\oint \Phi_\nu(\hat{k}, \hat{k}') d\Omega' = 1 = \oint \Phi_\nu(\hat{k}', \hat{k}) d\Omega. \quad (\text{I.23})$$

.11

- The full transfer equation in a continuous medium is thus:

$$\frac{dI_\nu}{ds} = j_\nu^{\text{abs}} - \alpha_\nu^{\text{abs}} I_\nu - \alpha_\nu^{\text{sca}} I_\nu + \underbrace{\alpha_\nu^{\text{sca}} \oint \Phi_\nu(\hat{k}, \hat{k}') I_\nu(\hat{k}') d\Omega'}_{j_\nu^{\text{scat}}}, \quad (\text{I.24})$$

with $\hat{k} = \hat{s}$, and where we have added the superscript ‘abs’ to the emission coefficient in the absence of scattering.

.12

- In thermodynamical equilibrium, $I_\nu(\hat{k}, \vec{x}) = B_\nu(T)$ and since the temperature field must be uniform, $\frac{dI_\nu}{ds} = 0$. Since the phase function is normalized, we have Kirchoff's law:

$$j_\nu^{\text{abs}} = \alpha_\nu^{\text{abs}} B_\nu. \quad (\text{I.25})$$

- Note that in Eq. I.24, the term highlighted as j_ν^{sca} effectively enters as an emission coefficient, i.e. it is not proportional to I_ν .
- Grouping together terms, Eq. I.24 can also be written

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu, \quad (\text{I.26})$$

with $j_\nu = j_\nu^{\text{abs}} + j_\nu^{\text{sca}}$ and $\alpha_\nu = \alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}$.

.13

I.2.4 Dicrete medium

- In a discrete medium the coupling of radiation and matter occurs through radiative transitions, each of which bears a line-profile $\phi(\nu)$ which is narrowly centred on the transition frequency ν_0 .
- The rates of radiative transitions can be described using the Einstein coefficients:
 - A_{ji} : probably of spontaneous radiative decay per unit time per absorber (units for A are s^{-1}).
 - $B_{ij}\bar{J}$: probably of absorption per unit time per absorber, with $\bar{J} = \int d\nu \phi(\nu) J_\nu$ is the frequency-averaged mean specific intensity (so that MKS units of B are $s^{-1}/(\text{Jy sr}^{-1})$).
 - $B_{ji}\bar{J}$: probably of stimulated emission per unit time.

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- The line profiles $\phi(\nu)$ are described with Lorentzians, and their width (or the line-widths), $\Delta\nu$, are directly connected to the spontaneous decay rate via the time-energy uncertainty principle, $\Delta E \Delta t \geq \hbar/2$:

$$\Delta t \sim \frac{1}{A_{ji}} \sim \frac{1}{\Delta\nu}. \quad (\text{I.27})$$

- Detailed balance connects the Einstein coefficients:

$$g_i B_{ij} = g_j B_{ji}, \quad (\text{I.28})$$

and

$$A_{ji} = \frac{2h\nu_{ji}^3}{c^2} B_{ji}. \quad (\text{I.29})$$

.15

- In a discrete and isotropic medium, the RT equation in its compact form (Eq.I.26) is also valid, and the emission and absorption coefficients are directly connected to the Einstein coefficients. In this case the phase function is constant, $\Phi_\nu(\hat{k}, \hat{k}') = 1/(4\pi)$.
- The emission coefficient is

$$j_\nu = \frac{h\nu_o}{4\pi} n_j A_{ji} \phi(\nu). \quad (\text{I.30})$$

This coefficient includes what would be called ‘scattering’ for the lines: a de-excitation following the absorption of the same photon.

- The absorption coefficient is

$$\alpha_\nu = \frac{h\nu_o}{4\pi} (n_i B_{ij} - n_j B_{ji}) \phi(\nu). \quad (\text{I.31})$$

Note that stimulated emission acts as negative absorption.

.16

I.3 Opacity and optical depth

I.3.1 Optical depth

- It is customary to use the opacities κ rather than the absorption coefficients α , with $\alpha = \kappa\rho$, where ρ is the mass density.
- We also define the optical depths along a ray,

$$\tau_\nu^{\text{abs}} = \int ds \alpha^{\text{abs}} = \int ds \rho \kappa^{\text{abs}}, \quad (\text{I.32})$$

and

$$\tau_\nu^{\text{sca}} = \int ds \alpha^{\text{sca}} = \int ds \rho \kappa^{\text{sca}}. \quad (\text{I.33})$$

.17

I.3.2 Mean free path

- The exponential decay in the pure absorption case suggests that $\exp(-\tau_\nu^{\text{abs}})$ is the probability that a photon survives the optical depth τ_ν^{abs} without absorption. This allows us to estimate the mean optical depth travelled by all photons:

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu \exp(-\tau_\nu) d\tau_\nu = 1. \quad (\text{I.34})$$

- The photon mean free path l_ν is the corresponding distance, i.e.

$$l_\nu^{\text{abs}} = \frac{1}{\alpha_\nu^{\text{abs}}} = \frac{1}{n\sigma_\nu^{\text{abs}}}, \quad (\text{I.35})$$

and similarly for scattering.

.18

I.3.3 Source Function

- Changing variables to $d\tau_\nu = (\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}})ds$, the full transfer equation can be written as

$$\frac{dI_\nu}{d\tau_\nu} = \underbrace{\frac{j_\nu^{\text{abs}} + j_\nu^{\text{sca}}}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}}}_{S_\nu} - I_\nu, \quad (\text{I.36})$$

where we have defined the Source Function S_ν .

- In the absence of scattering, i.e. if $\alpha_\nu^{\text{sca}} = 0$, then $S_\nu = B_\nu$.
- For an isotropic phase function, i.e. if $\Phi_\nu(\hat{k}, \hat{k}') = 1/(4\pi)$, the source function is

$$S_\nu = \frac{j_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}} J_\nu}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}} \quad (\text{I.37})$$

.19

I.4 Solutions of the transfer equation

I.4.1 The uniform slab

- In the widely used, but seldom exact, case of no scattering, the equation of radiative transfer Eq. I.36 is

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu, \quad (\text{I.38})$$

where $S_\nu = j_\nu^{\text{abs}}/\alpha_\nu^{\text{abs}}$. for thermal emission, $S_\nu = B_\nu$.

- It is straightforward to show that the solution to Eq. I.38 is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau'_\nu e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu). \quad (\text{I.39})$$

- For the uniform slab, with constant S_ν ,

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}). \quad (\text{I.40})$$

.20

I.4.2 Random walks

- If a medium is optically thick in scattering we expect a photon to undergo a random walk before its absorption or its escape from the medium. The net displacement is $\vec{R} = \sum_i \vec{s}_i$, and the dispersion in displacement is $\sigma(\vec{R}) \equiv (\langle \|\vec{R}\|^2 \rangle)^{1/2} = \sqrt{N}(\langle \|\vec{s}\|^2 \rangle)^{1/2}$. With $\langle \|\vec{s}\|^2 \rangle^{1/2} \sim l$, the photon mean free path, we have $\sigma(R) \sim \sqrt{N}l$.

- The probability of absorption in each step is

$$\epsilon = \frac{\alpha_\nu^{\text{abs}}}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}}. \quad (\text{I.41})$$

Since all steps are statistically independent, the probability of absorption for N steps is simply $N\epsilon_\nu$, and the average number of steps before absorption is $1/\epsilon_\nu$.

- Therefore, for an infinite medium, $\sigma(R) = l/\sqrt{\epsilon_\nu}$, which is sometimes called the ‘thermalization length’: if the medium is larger than $\sigma(R)$, then it is said to be effectively thick as most photons will be absorbed within the medium.

.21

I.4.3 Rosseland approximation

- Next to the idealization of full thermodynamical equilibrium, we can explore the consequences of a ‘local’ thermodynamical equilibrium, such that the deviations of I_ν from $B_\nu(T)$ are small.
- The transfer equation for isotropic scattering can be written as

$$\frac{dI_\nu}{ds} = -\overbrace{(\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}})}^{\alpha_\nu}(I_\nu - S_\nu), \quad (\text{I.42})$$

and in steady state, $\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial s} = \hat{s} \cdot \vec{\nabla} I_\nu$.

- For thermal radiative processes, $S_\nu = B_\nu$, so

$$I_\nu = B_\nu - \underbrace{\frac{1}{\alpha_\nu} \frac{\partial I_\nu}{\partial s}}_{\text{1st order correction}}. \quad (\text{I.43})$$

- Since the first order correction is expected to be small, we approximate $\frac{\partial I_\nu}{\partial s} \approx \frac{\partial B_\nu}{\partial s}$, and

$$I_\nu \approx B_\nu - \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial s}. \quad (\text{I.44})$$

.22

- For the flux $\vec{F}_\nu = \oint I_\nu \hat{k} d\Omega$ we integrate over solid angle. Concretely,

$$\vec{F}_\nu \cdot \hat{z} \equiv F_\nu = \oint \hat{k} \cdot \hat{z} I_\nu(\Omega) d\Omega = \oint \cos(\theta) I_\nu(\Omega) d\Omega. \quad (\text{I.45})$$

- For an application we need to rewrite Eq. I.44 in terms of $\frac{\partial}{\partial z}$:

$$\frac{\partial B_\nu}{\partial s} = \hat{s} \cdot \vec{\nabla} B_\nu = \mu \frac{\partial B_\nu}{\partial z}, \quad (\text{I.46})$$

in plane-parallel geometry.

- Direct integration of Eq. I.44 then yields

$$F_\nu = -\frac{4\pi}{3\alpha_\nu} \frac{\partial B_\nu}{\partial z}. \quad (\text{I.47})$$

.23

- The need for a plane-parallel approximation (required in Eq. I.46) can be avoided. We start by defining the angular moments of I_ν :

$$\begin{Bmatrix} cu_\nu \\ \vec{F}_\nu \\ c\mathbf{P}_\nu \end{Bmatrix} \equiv \oint \begin{Bmatrix} 1 \\ \hat{k} \\ \hat{k}\hat{k} \end{Bmatrix} I_\nu(\hat{k}) d\Omega. \quad (\text{I.48})$$

- Next we rewrite the RT equation as

$$\hat{k} \cdot \vec{\nabla} I_\nu = j_\nu^{\text{abs}} - \alpha_\nu I_\nu + \alpha_\nu^{\text{sca}} \oint \Phi(\hat{k}, \hat{k}') I_\nu(\hat{k}') d\Omega'. \quad (\text{I.49})$$

- We then evaluate $\oint d\Omega \hat{k} I_\nu$ (Eq. I.49):

$$\begin{aligned} \vec{\nabla} \cdot \oint \hat{k} \hat{k} I_\nu d\Omega &= \underbrace{\oint d\Omega \hat{k} j_\nu^{\text{abs}}}_{=0} - \alpha_\nu \vec{F}_\nu + \\ &\quad \alpha_\nu^{\text{sca}} \oint d\Omega' I_\nu(\hat{k}') \underbrace{\oint d\Omega \hat{k} \Phi(\hat{k}, \hat{k}')}_{=0, \text{ if } \Phi(-\hat{k}, \hat{k}') = \Phi(\hat{k}, \hat{k}')} . \end{aligned} \quad (\text{I.50})$$

- In Eq. I.50 we used forward-backward scattering symmetry, i.e. that $\Phi(-\hat{k}, \hat{k}') = \Phi(\hat{k}, \hat{k}')$, which is true for atomic processes (but not for large dust grains).
- We thus have

$$c\vec{\nabla} \cdot \mathbf{P} = -\alpha_\nu \vec{F}_\nu. \quad (\text{I.51})$$

- We now apply local thermodynamical equilibrium, so that $I_\nu(\vec{x}) \approx B_\nu(T(\vec{x}))$:

$$\mathbf{P}_{ij} \approx \oint d\Omega k_i k_j \frac{B_\nu}{c} = \frac{4\pi}{3c} B_\nu(T) \delta_{ij}. \quad (\text{I.52})$$

- Replacing into Eq. I.51,

$$F_\nu = \vec{F}_\nu \Big|_z \approx -\frac{4\pi}{3\alpha_\nu} \frac{\partial B_\nu(T)}{\partial z}. \quad (\text{I.53})$$

- Integrating over frequency for the total radiative flux:

$$F(z) = \int_0^\infty d\nu F_\nu(z) = -\frac{4\pi}{3} \frac{\partial T}{\partial z} \int_0^\infty d\nu \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T}. \quad (\text{I.54})$$

- We introduce the Rosseland mean absorption coefficient,

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty d\nu \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T}}{\int_0^\infty d\nu \frac{\partial B_\nu}{\partial T}}, \quad (\text{I.55})$$

and note that $\int_0^\infty d\nu \frac{\partial B_\nu}{\partial T} = \frac{4}{\pi} \sigma T^3$.

- Finally, the flux in LTE is:

$$\vec{F} \cdot \hat{s} = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial s}. \quad (\text{I.56})$$

.26

- We can also write Eq. I.56 as

$$\vec{F} \cdot \hat{s} = -\frac{4\sigma}{3\alpha_R} \frac{\partial T^4}{\partial s}, \quad (\text{I.57})$$

or

$$\vec{F} \cdot \hat{s} = -\frac{4c}{3\alpha_R} \frac{\partial u}{\partial s}, \quad (\text{I.58})$$

where u is the total radiative energy density.

- It is interesting to note the similitude between Rosseland's approximation and Fick's law for diffusion:

$$\text{flux} \sim -D \vec{\nabla} \text{ quantity being diffused}, \quad (\text{I.59})$$

in this case $D = \frac{1}{3}cl$ with $l \equiv 1/\alpha_R$ would be an effective mean free path.

.27

- A problem with Rosseland's approximation is that the diffusion coefficient D diverges as $\rho \rightarrow 0$. In order to extend its domain of validity, the 'lambda-controlled' diffusion has been proposed:

$$\vec{F} \cdot \hat{s} = -\Lambda \frac{4c}{3\alpha_R} \frac{\partial u}{\partial s}, \quad (\text{I.60})$$

For instance

$$\Lambda = \frac{2 + R}{6 + 3R + R^2}, \quad (\text{I.61})$$

with

$$R = \frac{1}{\alpha_R} \frac{\|\vec{\nabla} u\|}{u}, \quad (\text{I.62})$$

see Casassus et al. (2019, MNRAS, 486L, 58; and references therein) for an application.

.28

I.4.4 Eddington approximation

- In plane-parallel geometry the radiation moments introduced in Sec. I.1.2 and in Eq. I.48 can be summarised as

$$\begin{pmatrix} J_\nu \\ H_\nu \\ K_\nu \end{pmatrix} \equiv \oint_{-1}^1 \frac{1}{2} \begin{pmatrix} 1 \\ \mu \\ \mu^2 \end{pmatrix} I_\nu(\mu) d\mu. \quad (\text{I.63})$$

- By comparison with Eq. I.48, we have $J_\nu = c \frac{u_\nu}{4\pi}$, $H_\nu = \frac{\vec{F}_\nu|_z}{4\pi}$, $K_\nu = c \frac{\mathbf{P}_\nu|_{zz}}{4\pi}$.
- In the treatment that leads to the Eddington approximation, we assume that $I_\nu(\mu)$ is nearly isotropic and we expand to first order in μ :

$$I_\nu = a_\nu + b_\nu \mu, \quad (\text{I.64})$$

i.e. similar to a dipole term $b\mu$ modifying the monopole term a .

.29

- Direct evaluation of Eq. I.63 using Eq. I.64 leads to $J_\nu = a_\nu$, $H_\nu = \frac{b_\nu}{3}$ and $K_\nu = \frac{a_\nu}{3}$.
- We therefore have

$$K_\nu = \frac{1}{3} J_\nu, \quad (\text{I.65})$$

exactly as for thermal radiation, for which the radiation pressure is $p_\nu = \frac{1}{3} u_\nu$, but here in a more general case. Eq. I.65 is known as the Eddington approximation.

- The plane-parallel RT equation following from Eq. I.42 and Eq. I.46 is

$$\mu \frac{\partial I_\nu}{\partial z} = -\alpha_\nu (I_\nu + S_\nu), \quad (\text{I.66})$$

or

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = -(I_\nu + S_\nu), \quad (\text{I.67})$$

.30

- We now take $\int d\mu$ (Eq. I.67),

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu, \quad (\text{I.68})$$

and $\int d\mu \mu$ (Eq. I.67),

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu. \quad (\text{I.69})$$

- Using the Eddington approximation (Eq. I.65), we obtain an equation for J_ν as a function of S_ν :

$$\frac{1}{3} \frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = J_\nu - S_\nu, \quad (\text{I.70})$$

- In terms of the albedo $\omega_\nu = 1 - \epsilon_\nu$ (see Eq. I.41),

$$\frac{1}{3} \frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \epsilon(\nu)(J_\nu - B_\nu), \quad (\text{I.71})$$

which, if the medium properties are known, is a 2nd-order equation for $J_\nu(\tau_\nu)$. If it is possible to solve Eq. I.71 for $J_\nu(\tau_\nu)$, then we have I_ν by integration of Eq. I.67.

.31

I.4.5 Two-stream approximation

- We have seen that the Eddington approximation leads to a 2nd-order equation for J_ν (Eq. I.71) which approximately solves the RT problem in plane-parallel geometry. However boundary conditions are required to solve Eq. I.71. The two-stream approximations provides a set of such boundary conditions.
- In the two stream approximation, we assume that the whole specific intensity field is restricted to only two directions with $\mu = \pm \frac{1}{\sqrt{3}}$:

$$I_\nu(\mu) = I_\nu^+ \delta\left(\mu - \frac{1}{\sqrt{3}}\right) + I_\nu^- \delta\left(\mu + \frac{1}{\sqrt{3}}\right). \quad (\text{I.72})$$

- Direct evaluation yields

$$\begin{Bmatrix} J_\nu \\ H_\nu \\ K_\nu \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(I^+ + I^-) \\ \frac{1}{2\sqrt{3}}(I^+ - I^-) \\ \frac{1}{6}(I^+ + I^-) \end{Bmatrix}. \quad (\text{I.73})$$

.32

- We see that in the two-stream approximation, $K_\nu = \frac{1}{3}J_\nu$, which is the Eddington approximation. The transfer equation Eq. I.67 also applies to the two streams, so taking moments and applying the Eddington approximation also yields Eq. I.71. Thus, the two streams also satisfy Eq. I.71.
- Using the Eddington approx. in Eq. I.69,

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3} \frac{\partial J_\nu}{\partial \tau_\nu}, \quad (\text{I.74})$$

the moments in Eq. I.73 can be rewritten as

$$I^+ = J_\nu + \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu}, \quad (\text{I.75})$$

$$I^- = J_\nu - \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu}. \quad (\text{I.76})$$

$$(\text{I.77})$$

.33

- To finally write the required boundary conditions, we focus on a slab geometry from $\tau_\nu = 0$ to $\tau_\nu = \tau_\circ$. Clearly, no radiation will come from $\tau_\nu = +\infty$, so $I^-(\tau_\circ) = 0$. Likewise, $I^+(0) = 0$. Therefore, the required boundaries are:

$$\frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu} = J_\nu, \text{ at } \tau_\nu = \tau_\circ \quad (\text{I.78})$$

$$\frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu} = -J_\nu \text{ at } \tau_\nu = 0. \quad (\text{I.79})$$

Part II

Electromagnetic wave propagation

II.1 Electromagnetic waves

II.1.1 Maxwell Equations

- In the MKS system (or S.I.), the equations of electrodynamics are, :

$$\vec{\nabla} \cdot \vec{D} = \rho, \quad (\text{II.1})$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (\text{II.2})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (\text{II.3})$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (\text{II.4})$$

- For linear media, $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.
- In vacuum, $\epsilon = \epsilon_0$ and $\mu = \mu_0$.

.35

II.1.2 Electrodynamic potentials

- Since $\vec{\nabla} \cdot \vec{B} = 0$, we have

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (\text{II.5})$$

- For $\vec{E}$, we use Eq. II.5 and Eq. II.3:

$$\vec{\nabla} \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right)}_{\equiv -\vec{\nabla} \Phi} = 0, \Rightarrow$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}. \quad (\text{II.6})$$

.36

II.1.3 Wave equations

- We want to write equations that determine the electrodynamic potentials $\vec{A}$ and Φ .

- Using the Maxwell Equations in vacuum to connect directly with $\vec{E}$ and $\vec{B}$, we have

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \Phi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = \frac{\rho}{\epsilon_0}, \quad (\text{II.7})$$

$$\begin{aligned} \vec{\nabla} \times \frac{1}{\mu_0} \vec{B} &= \vec{J} + \frac{1}{\epsilon_0} \frac{\partial \vec{E}}{\partial t} \Rightarrow \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \underbrace{\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right)}_{\text{term for the Lorentz condition}} &= -\mu_0 \vec{J}. \end{aligned} \quad (\text{II.8})$$

- If the term highlighted in Eq. II.8 is null, which is called the *Lorentz Condition*,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0, \quad (\text{II.9})$$

then we recover the wave equation for the potentials:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad (\text{II.10})$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}. \quad (\text{II.11})$$

- To fulfill the Lorentz condition, we use the freedom of gauge:

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla} \Lambda, \quad (\text{II.12})$$

which leaves invariant $\vec{B} = \vec{\nabla} \times \vec{A}$.

- To also preserve $\vec{E} = -\vec{\nabla} \Phi - \partial \vec{A} / \partial t$, it is necessary that

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \quad (\text{II.13})$$

- If $\vec{A}$ and Φ both fulfill the general potential equations (Eqs. II.8 and II.7), but do not fulfill the Lorentz condition, then we can search for $\Lambda(\vec{x}, t)$ so that $\vec{A}'$ and Φ' do satisfy the Lorentz condition.
- Injecting Eqs. II.12 and II.13 in Eq. II.9, we reach an equation for $\Lambda(\vec{x}, t)$:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0, \quad (\text{II.14})$$

which is essentially a wave equation with a source term, i.e. exactly the type of equations that we will propose solutions for.

- Independently of the Lorentz condition, we can manipulate the Maxwell equations to reach (tarea):

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \left(-\vec{\nabla} \rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \quad (\text{II.15})$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J}, \quad (\text{II.16})$$

which are both wave equations with source terms.

- Away from the sources, i.e. in vacuum, both equations become the homogeneous wave equation.

.40

II.1.4 Poynting's theorem

- The power exerted by the electromagnetic force $\vec{F} = q\vec{v} \times \vec{B} + q\vec{E}$ on a single charge q with velocity $\vec{v}$ is $\vec{v} \cdot \vec{F} = q\vec{v} \cdot \vec{E}$.
- The power exerted on the charge density distribution ρ and on the current density distribution $\vec{J} = \rho\vec{v}$ inside a volume $d\mathcal{V}$ is thus

$$dP = \vec{J} \cdot \vec{E} d\mathcal{V}$$

- The total power exerted by the $(\vec{E}, \vec{B})$ field on the charges inside a volume $\mathcal{V}$ is

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x. \quad (\text{II.17})$$

.41

- We want to connect P with the energy stored in the fields. Using the Ampère-Maxwell equation (Eq. II.4) we solve for $\vec{J}$, and following standard handling (tarea),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \quad (\text{II.18})$$

- Now with the induction law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Eq. II.3),

$$P = \int_{\mathcal{V}} \left[-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x. \quad (\text{II.19})$$

.42

- Remembering that for a linear medium $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})$, and $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$, we reach

$$P = \int_{\mathcal{V}} \vec{J} \cdot \vec{E} d^3x = - \int_{\mathcal{V}} \left[\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right] d^3x, \quad (\text{II.20})$$

where we recognize

$$u = \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \vec{H}, \quad (\text{II.21})$$

and

$$\vec{S} = \vec{E} \times \vec{H}. \quad (\text{II.22})$$

- For any volume $\mathcal{V}$, we conclude that

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}. \quad (\text{II.23})$$

.43

- In the same way as for energy conservation, Eq. II.23, we can also write the equation for the conservation of linear momentum. Newton's 2nd law for the variation of linear momentum $\delta \vec{p}_{\text{mec}}$ inside a volume $\delta \mathcal{V}$ is:

$$\frac{d \delta \vec{p}_{\text{mec}}}{dt} = \rho \vec{E} \delta \mathcal{V} + \rho \vec{v} \times \vec{B} \delta \mathcal{V}. \quad (\text{II.24})$$

- In total,

$$\frac{d \vec{p}_{\text{mec}}}{dt} = \int_{\mathcal{V}} d^3x (\rho \vec{E} + \rho \vec{v} \times \vec{B}). \quad (\text{II.25})$$

.44

- Using Maxwell's equation to replace ρ and $\vec{J}$, we reach:

$$\frac{d}{dt} (\vec{p}_{\text{mec}} + \vec{p}_{\text{fields}})|_i = \sum_j \int_{\mathcal{V}} d^3x \frac{\partial T_{ij}}{\partial x_j}, \quad (\text{II.26})$$

with the following notations:

$$\vec{p}_{\text{fields}} = \int \epsilon_0 (\vec{E} \times \vec{B}) d^3x = \frac{1}{c^2} \int d^3x \vec{S}, \quad (\text{II.27})$$

which we associate to the momentum in the fields since it fulfills a similar role as $\vec{p}_{\text{mec}}$, and

$$T_{ij} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} (\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B}) \delta_{ij} \right], \quad (\text{II.28})$$

which is the tensor of electromagnetic tensions.

.45

- For each component i the integrand of Eq. II.26 (involving T_{ij}) can be seen as a divergence, so

$$\frac{d}{dt}(\vec{p}_{\text{mec}} + \vec{p}_{\text{fields}})|_i = \oint_S \sum_j T_{ij} n_j d\mathcal{A}, \quad (\text{II.29})$$

where we recognize a flux integral over the surface bounding the volume $\mathcal{V}$.

.46

II.2 Wave propagation in vacuum

II.2.1 Spectral decomposition

- In the absence of sources, if we decompose

$$\vec{E}(\vec{x}, t) = \frac{1}{2\pi} \int d\omega \vec{E}(\vec{x}, \omega) e^{i\omega t}, \quad (\text{II.30})$$

the Maxwell equations yield

$$(\nabla^2 + \mu\epsilon\omega^2) \begin{Bmatrix} \vec{E} \\ \vec{B} \end{Bmatrix} = 0. \quad (\text{II.31})$$

- If ϵ and μ are both real, the solutions are $e^{\pm i k x}$, with $k = \sqrt{\mu\epsilon}\omega$
- We define the phase velocity $v_\phi = \frac{\omega}{k} = \frac{c}{n}$, where $n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$ is the refraction index.
- In general,

$$\begin{Bmatrix} E_i \\ B_i \end{Bmatrix} = \frac{1}{2\pi} \int d\omega \begin{Bmatrix} \mathcal{E}_i \\ \mathcal{B}_i \end{Bmatrix} e^{\pm i \vec{k} \cdot \vec{x} - i\omega t}, \quad (\text{II.32})$$

.47

- We recognize d'Alembert's solution for the wave equation,

$$\begin{Bmatrix} E_i \\ B_i \end{Bmatrix} = \frac{1}{2\pi} \int d\omega \begin{Bmatrix} \mathcal{E}_i \\ \mathcal{B}_i \end{Bmatrix} e^{\pm i k(\hat{n} \cdot \vec{x} - v_\phi t)}, \quad (\text{II.33})$$

where each component i has a form $f(\hat{n} \cdot \vec{x} - v_\phi t) + g(\hat{n} \cdot \vec{x} + v_\phi t)$, and where $\hat{n}$ is the direction of propagation.

- Using Maxwell's equations (**ta**rea), $\hat{n} \cdot \vec{\mathcal{E}} = 0$, $\hat{n} \cdot \vec{\mathcal{B}} = 0$ and $\vec{\mathcal{B}} = \frac{n}{c} \hat{n} \times \vec{\mathcal{E}}$.

.48

- For harmonic fields it is customary to use complex notation (because of the spectral decomposition), so that $\vec{S} = \Re(\vec{E}) \times \Re(\vec{H})$.
- In general for products of the type

$$\Re(ae^{-i\omega t})\Re(be^{-i\omega t}) = \frac{1}{2}\Re(a^*b + abe^{-2i\omega t}), \quad (\text{II.34})$$

it is also customary to take time averages $\langle(\cdots)\rangle_T = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\cdots) dt$, and (tarea)

$$\langle\Re(ae^{-i\omega t})\Re(be^{-i\omega t})\rangle = \frac{1}{2}\Re(a^*b). \quad (\text{II.35})$$

.49

- We therefore have

$$\langle\vec{S}\rangle = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\mathcal{E}|^2\hat{n}. \quad (\text{II.36})$$

And similarly,

$$\langle u \rangle = \frac{1}{4}(\epsilon\vec{E} \cdot \vec{E}^* + \frac{1}{\mu}\vec{B} \cdot \vec{B}^*) = \frac{\epsilon}{2}|\mathcal{E}|^2. \quad (\text{II.37})$$

- Finally, $\langle\vec{S}\rangle = v_\phi u \hat{n}$.

.50

II.2.2 Connection with radiative transfer

- We can now see that the concept of rays associated to the radiative transfer equation, which describes the transport of radiation in a straight line, is connected to the idea of a plane monochromatic wave with direction of propagation $\vec{k}$.
- For a plane wave $\langle\vec{S}\rangle = v_\phi u \hat{k}$ is the flux of energy in direction $\hat{k}$.
- In radiative transfer notation, the flux density in direction $k_\circ$ would be

$$F_\nu(\vec{x}) = \int d\Omega I_\nu(\hat{k}, \vec{x}) \hat{k} \cdot \hat{k}_\circ. \quad (\text{II.38})$$

- Therefore the specific intensity field for a monochromatic plane wave is

$$I_\nu(\hat{k}) = \|\vec{S}_\nu\| \delta(\hat{k} - \hat{k}_\circ). \quad (\text{II.39})$$

.51

II.2.3 Polarization

- In summary, the electric field of a monochromatic wave can be decomposed in two linearly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{e}_1 E_1 + \hat{e}_2 E_2) e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad (\text{II.40})$$

whose total describes, in general, an elliptically polarized wave.

- With a change of vectorial basis to $\hat{e}_{\pm} = \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$, we can also decompose $\vec{E}$ in two circularly polarized waves,

$$\vec{E}(\vec{x}, t) = (\hat{e}_+ E_+ + \hat{e}_- E_-) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (\text{II.41})$$

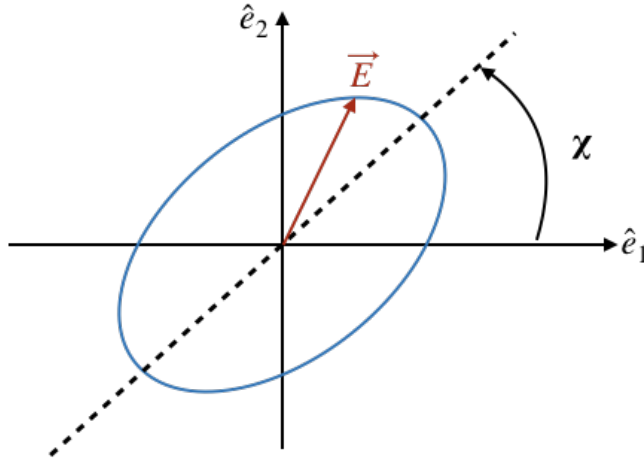
.52

- With the notation

$$\begin{aligned} E_1 &= \mathcal{E}_1 e^{i\phi_1}, & E_2 &= \mathcal{E}_2 e^{i\phi_2}, \\ E_+ &= \mathcal{E}_+ e^{i\phi_+}, & E_- &= \mathcal{E}_- e^{i\phi_-}, \end{aligned}$$

we have

$$\begin{cases} \text{linear polarization : } \phi_2 - \phi_1 = 0. \\ \text{circular polarization : } |\phi_2 - \phi_1| = \frac{\pi}{2} \text{ and } \mathcal{E}_2 = \mathcal{E}_1. \\ \text{the general case is elliptical, with : } \tan(\chi) = \frac{\mathcal{E}_1 \cos(\phi_1)}{\mathcal{E}_2 \cos(\phi_2)}. \end{cases}$$



.53

- It is customary to use the Stokes parameters to characterize the polarization state of **monochromatic light**:

$$\begin{aligned} I &= E_1 E_1^* + E_2 E_2^* = \mathcal{E}_1^2 + \mathcal{E}_2^2, \\ Q &= E_1 E_1^* - E_2 E_2^* = \mathcal{E}_1^2 - \mathcal{E}_2^2, \\ U &= E_1 E_2^* - E_2 E_1^* = 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_2 - \phi_1), \\ V &= i(E_1 E_2^* - E_2 E_1^*) = 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_2 - \phi_1). \end{aligned} \quad (\text{II.42})$$

- We see that Stokes I (the total “radiance”) is $I \propto |\vec{S}|$, Q and U measure linear polarization, while V measure circular polarization. In order to make this obvious it is best to use mental experiments with polarizers that select specific types of polarization (see class).
- For a strictly monochromatic wave, it follows that

$$I^2 = Q^2 + U^2 + V^2. \quad (\text{II.43})$$

.54

II.2.4 Quasi-monochromatic waves

- In order to obtain $\vec{E}(\vec{x}, \omega)$, we need to know $\vec{E}(t)$ for all t , since

$$\vec{E}(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{x}, t) e^{i\omega t} dt. \quad (\text{II.44})$$

- So in practice, we treat E_1 and E_2 as random variables, i.e. for a wave in vacuum, described by Eq. II.40,

$$\vec{E}(\vec{x}, t) = (E_1(t)\hat{e}_1 + E_2(t)\hat{e}_2) e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \quad (\text{II.45})$$

Alternatively we can also replace the time dependence in Eq. II.45 with a probability density, which itself may depend on time.

- To fix ideas, let's remember that $\Delta t \Delta \omega = 1$ for Gaussian spectra, where Δt is the ‘coherence time’, and $\Delta \omega$ is the ‘bandwidth’ of the quasi-monochromatic wave.

.55

- In order to measure the Stokes parameters, we need averages of the kind

$$\langle E_1 E_2^* \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int dt E_1(t) E_2^*(t) dt. \quad (\text{II.46})$$

- We therefore have

$$\begin{aligned} \langle Q^2 \rangle + \langle U^2 \rangle + \langle V^2 \rangle &= \langle I^2 \rangle - \\ 4(\langle \mathcal{E}_1^2 \rangle \langle \mathcal{E}_2^2 \rangle - \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \cos^2(\phi_2 - \phi_1) \rangle + \langle \mathcal{E}_1^2 \mathcal{E}_2^2 \sin^2(\phi_2 - \phi_1) \rangle), \end{aligned} \quad (\text{II.47})$$

and, by Schwartz' inequality ($\langle \|a\|^2 \rangle \langle \|b\|^2 \rangle \geq \langle a \cdot b \rangle^2$),

$$I^2 \geq Q^2 + U^2 + V^2. \quad (\text{II.48})$$

.56

- For a wave with a single and constant elliptical polarization state, the equality holds in Eq. II.48.
- On the other hand, for a completely unpolarized wave, $Q = U = V = 0$.
- The Stokes parameters are additive. Proof: consider a sum of N different waves

$$\vec{E} = \sum_{k=1}^N \vec{E}^k = \sum (\hat{e}_1 E_1^k + \hat{e}_2 E_2^k) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (\text{II.49})$$

Because each $E_i^k(t)$ is statistically independent, $\langle E_i^k E_j^{l*} \rangle = \delta_{kl} \langle E_i^k E_j^{k*} \rangle$, and

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \sum_k \begin{pmatrix} I_k \\ Q_k \\ U_k \\ V_k \end{pmatrix}. \quad (\text{II.50})$$

- We can therefore decompose an arbitrary set of Stokes parameters in

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \overbrace{\begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\text{unpol}} + \overbrace{\begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}}^{\text{pol}}. \quad (\text{II.51})$$

- The first term ‘unpol’ is completely unpolarized since $Q = U = V = 0$, while the second term ‘pol’ is completely polarized since it satisfies $I^2 = Q^2 + U^2 + V^2$ (Eq. II.43).

- The total polarized intensity of a wave train is thus be $I^{\text{pol}} = \sqrt{Q^2 + U^2 + V^2}$.
- We define the polarization fraction as

$$\Pi = \frac{I^{\text{pol}}}{I}. \quad (\text{II.52})$$

II.3 Wave propagation in a medium

II.3.1 Constitutive equations

- Each monochromatic component of the field $\vec{E}, \vec{B}$ must fulfill the following constitutive relations:

$$\begin{aligned}\vec{P} &= \epsilon_o \chi \vec{E} &\longrightarrow & \vec{P}(\omega) = \epsilon_o \chi(\omega) \vec{E}(\omega), \\ \vec{B} &= \mu \vec{H} &\longrightarrow & \vec{B}(\omega) = \mu(\omega) \vec{H}(\omega), \\ \vec{J} &= \sigma \vec{E} &\longrightarrow & \vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega),\end{aligned}\tag{II.53}$$

in which we have added Ohm's law.

- We note that Hermitian symmetry, $\chi(-\omega) = \chi^*(\omega)$, is required for $\chi(t) = \frac{1}{2\pi} \int d\omega \chi(\omega) \exp(-i\omega t)$ to be real, and similarly for μ and σ .

.60

- The Fourier convolution theorem states that if $X(\omega) = Y(\omega)Z(\omega)$, then

$$X(t) = \int_{-\infty}^{\infty} Y(t-t')Z(t')dt',\tag{II.54}$$

where $Y(t) = \frac{1}{2\pi} \int d\omega Y(\omega) \exp(-i\omega t)$, etc..

- Applying the convolution theorem to χ (for example),

$$\begin{aligned}\vec{P}(t) &= \int_{-\infty}^{\infty} G(t-t')\vec{E}(t')dt', \text{ with} \\ G(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_o \chi(\omega) e^{-i\omega t} d\omega.\end{aligned}\tag{II.55}$$

- We see that $\vec{P}(t)$ depends on the history of $\vec{E}(t')$, which bears physical sense only in the past, for $t' < t$, so $G(t) = 0$ if $t < 0$. We will use this property in the next section.

.61

- Note that $\vec{P}(t)$ is no longer proportional to the applied field $\vec{E}(t)$, except for harmonic fields. To see this we write the monochromatic wave as

$$\vec{E}(t) = \vec{A} \cos(\omega_o t) + \vec{B} \sin(\omega_o t) = \Re(\vec{E}_c(t)),\tag{II.56}$$

with $\vec{E}_c = (\vec{A} - i\vec{B})(\cos(\omega_o t) + i \sin(\omega_o t))$.

- In the Fourier plane,

$$\vec{E}(\omega) = \pi \left[(\vec{A} + i\vec{B})\delta(\omega - \omega_o) + (\vec{A} - i\vec{B})\delta(\omega + \omega_o) \right].\tag{II.57}$$

- We can evaluate

$$\begin{aligned}\vec{P}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_o \chi(\omega) \vec{E}(\omega) e^{-i\omega t} d\omega \\ &= \Re \left[\frac{\epsilon_o}{2} (\vec{A} - i\vec{B}) \chi(\omega_o) e^{-i\omega_o t} \right] = \Re[\vec{P}_c(t)],\end{aligned}\tag{II.58}$$

using $\chi(-\omega) = \chi^*(\omega)$, and where

$$\vec{P}_c(t) = \epsilon_o \chi(\omega_o) \vec{E}_c(t).\tag{II.59}$$

.62

- With the spectral decomposition of the constitutive relations we can rewrite the Maxwell equations in their harmonic versions. In the absence of free charges,

$$\begin{aligned}\vec{\nabla} \cdot \vec{E}(\omega) &= 0, & \vec{\nabla} \times \vec{E}(\omega) &= -i\omega\mu(\omega)\vec{H}(\omega), \\ \vec{\nabla} \cdot \vec{H}(\omega) &= 0, & \vec{\nabla} \times \vec{H}(\omega) &= -i\omega\epsilon(\omega)\vec{E}(\omega),\end{aligned}\tag{II.60}$$

where (tarea)

$$\epsilon(\omega) = \epsilon_o(1 + \chi(\omega)) + i\frac{\sigma(\omega)}{\omega}.\tag{II.61}$$

- Note that both susceptibility and conductivity contribute to the imaginary part of ϵ :

$$\Im(\epsilon) = \epsilon_o \Im(\chi) + \Re(\sigma/\omega).\tag{II.62}$$

.63

II.3.2 Kramers-Kronig relations

- From physical considerations we can anticipate that the induced $P(t)$ depends on the history of the applied field, or

$$\vec{P}(t) = \int_{-\infty}^{\infty} G(t, t') \vec{E}(t') dt'\tag{II.63}$$

(note difference with Eq. II.55).

- Let's assume that $\vec{E} = \delta(t - t_o) \vec{E}_o$. Then $\vec{P}(t) = G(t, t_o) \vec{E}_o$, and G is the polarization resulting from a delta-unitary electric field.
- If the properties of the medium do not change in time, $G(t, t_o) = G(t - t_o)$, and we recover Eq. II.55.
- Causality requires that $G(\tau) = 0$ if $\tau < 0$, so

$$\epsilon_o \chi(\omega) = \int_0^{\infty} dt G(t) e^{i\omega t}.\tag{II.64}$$

.64

- We extend Eq. II.64 to the complex plane with $\tilde{\omega} = \omega_R + i\omega_I$, where $\omega_I > 0$.

$$\epsilon_o \chi(\tilde{\omega}) = \int_0^\infty dt G(t) e^{i\tilde{\omega}t}. \quad (\text{II.65})$$

- If $\int_0^\infty |G(t)| dt$ converges, so does $\int_0^\infty G(t) e^{i\tilde{\omega}t} dt$, and $\chi(\tilde{\omega})$ is analytical in the superior $\mathbb{C}$ plane ($\omega_I > 0$).
- Therefore $\chi(\tilde{\omega})/(\tilde{\omega} - \omega)$ is analytical except in the pole $\tilde{\omega} = \omega$, where ω is a point along the real axis.
- We can apply the Kramers-Kronig theorem (proof: see Bohren & Huffman, Sec. 2.3.2), which gives

$$i\pi\chi(\omega) = P \int_{-\infty}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega, \quad (\text{II.66})$$

where P indicates Cauchy's 'principal value'

$$P \int_{-\infty}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega = \lim_{a \rightarrow 0} \left(\int_{-\infty}^{\omega-a} \frac{\chi(\Omega)}{\Omega + \omega} d\Omega + \int_{\omega+a}^\infty \frac{\chi(\Omega)}{\Omega + \omega} d\Omega \right). \quad (\text{II.67})$$

- Using that $\chi^*(\Omega) = \chi(-\Omega)$ we can restrict the integration to $\Omega > 0$, and use $\chi = \chi_R + i\chi_I$ to rewrite Eq. II.67:

$$\chi_R(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\Omega \chi_I(\Omega)}{\Omega^2 - \omega^2} d\Omega, \quad (\text{II.68})$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\chi_R(\Omega)}{\Omega^2 - \omega^2} d\Omega. \quad (\text{II.69})$$

- Similar relationships exists for μ y σ .

II.3.3 Monochromatic waves

- We now extend the monochromatic waves to homogeneous media. We inject

$$\vec{E}_c = \vec{E}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \text{ and } \vec{H}_c = \vec{H}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad (\text{II.70})$$

into Maxwell's equations.

- Allowing for $\vec{k} \in \mathbb{C}$, $\vec{k} = \underbrace{(k_R + ik_I)}_k \hat{e}$,

$$\vec{E}_c = \vec{E}_o e^{-\vec{k}_I \cdot \vec{x}} e^{i(\vec{k}_R \cdot \vec{x} - \omega t)}. \quad (\text{II.71})$$

- The harmonic Maxwell equations (Eqs. II.60) yield:

$$\begin{aligned}\vec{k} \cdot \vec{E}_o &= 0 & \vec{k} \cdot \vec{H}_o(\omega) &= 0 \\ \vec{k} \times \vec{E}_o &= \omega\mu\vec{H}_o, & \vec{k} \times \vec{H}_o &= -\omega\epsilon\vec{E}_o.\end{aligned}\quad (\text{II.72})$$

- Eliminating $\vec{E}_o$ and $\vec{H}_o$ (for example taking $\vec{k} \times \vec{k} \times \vec{E}_o$, as in Bohren & Huffman 2.6), we obtain $\vec{k} \cdot \vec{k} = \omega^2\epsilon\mu$ (tarea) and

$$k_R^2 - k_I^2 + 2ik_I \cdot \vec{k}_R = \omega^2\epsilon\mu \quad (\text{tarea}). \quad (\text{II.73})$$

.67

- So, for a homogeneous wave (no free charges),

$$\vec{k} = \underbrace{(k_R + ik_I)}_k \hat{e},$$

and $k = \omega N/c$, where N is the complex refractive index,

$$N = c\sqrt{\epsilon\mu} = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}}.$$

- We set $N = n + i\kappa$, where n and κ are both $\in \mathbb{R}^+$.
- Eq. II.71 gives:

$$\vec{E}_c = \vec{E}_o e^{-\frac{2\pi}{\lambda}\kappa z} e^{i(\frac{2\pi n z}{\lambda} - i\omega t)}. \quad (\text{II.74})$$

- $\Rightarrow$ the imaginary part of N corresponds to absorption.

.68

- We can apply the Kramers-Kronig relations to $(N(\omega) - 1)$ (the -1 is motivated by $\lim_{\omega \rightarrow \infty} N(\omega) = 1$):

$$\begin{aligned}n(\omega) - 1 &= \frac{2}{\pi} P \int_0^\infty \frac{\Omega \kappa(\Omega)}{\Omega^2 - \omega^2} d\Omega \\ \kappa(\omega) &= -\frac{2\omega}{\pi} P \int_0^\infty \frac{n(\Omega)}{\Omega^2 - \omega^2} d\Omega\end{aligned}\quad (\text{II.75})$$

- We see that the absorption in a medium is also related to the real refractive index.

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Part III

Radiation

III.1 General solution of the wave equation

III.1.1 Green function for the wave equation

- This section follows from Jackson 3rd ed, Sec. 6.4.
- In order to determine $\vec{A}(\vec{x}, t)$ and $\Phi(\vec{x}, t)$, we need to solve the wave equation with source terms. For a generic field $\Psi(\vec{x}, t)$,

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t). \quad (\text{III.1})$$

- It is convenient to use the Fourier time-domain,

$$\psi(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(\vec{x}, \omega) e^{-i\omega t} d\omega, \quad (\text{III.2})$$

whose inverse is

$$\psi(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \psi(\vec{x}, t) e^{i\omega t} dt. \quad (\text{III.3})$$

- Injecting Ec. III.2 in Ec. III.1, we reach the Helmholtz equation:

$$(\nabla^2 + k^2) \Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega). \quad (\text{III.4})$$

- The Helmholtz equation Eq. III.4 is very similar to the Poisson equation, and we can anticipate the use of similar machinery in its solution. The Green function satisfies

$$(\nabla^2 + k^2) G_k(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x}, \vec{x}'). \quad (\text{III.5})$$

- Changing coordinates to a system with origin in $\vec{x}'$, we see that Eq. III.5 has spherical symmetry, and $G_k(\vec{x}, \vec{x}') = G_k(R)$, with $R = |\vec{R}|$ and $\vec{R} = \vec{x} - \vec{x}'$
- Eq. III.5 can thus be written as

$$\frac{1}{R} \frac{d^2}{dR^2} (R G_k) + k^2 G_k = -4\pi \delta(\vec{R}). \quad (\text{III.6})$$

- If $R \neq 0$, the solution to Eq. III.6 is $R G_k = A e^{ikR} + B e^{-ikR}$, where the constants A and B do not depend on k . In order to determine these constants, we use the case $k = 0$, i.e. Poisson, whose solution is $G_{k=0}(R) = 1/R$, $\rightarrow A + B = 1$.

- Thus the general solution to Eq. III.6 is

$$G_k(R) = AG_k^+(R) + BG_k^-(R), \text{ with } G_k^\pm = \frac{e^{\pm ikR}}{R} \quad (\text{III.7})$$

and $A + B = 1$.

- The values of A and B depend on the initial conditions, i.e. on the boundary conditions in time. To see this, we return to the time domain and we generalize Eq. III.5:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^\pm(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \quad (\text{III.8})$$

- Now, returning to the frequency domain ω , we generalize Eq. III.5 to

$$(\nabla^2 + k^2)G_k(\vec{x}, \vec{x}'; t') = -4\pi\delta(\vec{x} - \vec{x}')e^{i\omega t'}, \quad (\text{III.9})$$

with solution $G_k^\pm(R)e^{i\omega t'}$.

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- To return once more to the time domain, we use Eq. III.2, and

$$G^\pm(R; t, t') = G^\pm(R, \tau) = \frac{1}{2\pi} \int \frac{e^{\pm ikR - i\omega\tau}}{R} d\omega,$$

where $\tau = t - t'$.

- For a non-dispersive medium (one with $\omega/k = c$), we reach

$$G^\pm(\vec{x}, t; \vec{x}', t') = \frac{\delta\left(t' - \left[t \mp \frac{|\vec{x} - \vec{x}'|}{c}\right]\right)}{|\vec{x} - \vec{x}'|} \quad (\text{III.10})$$

- We apply this Green function to write the generic solutions of Ec. III.1:

$$\Psi^\pm(\vec{x}, t) = \int d^3x' dt' G^\pm(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'). \quad (\text{III.11})$$

.73

- The $+$ case corresponds to the retarded solution, with an entry or an initial condition ψ_{in} (valid before the sources f are activated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t < 0$):

$$\Psi^+(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t) + \int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (\text{III.12})$$

where we see that if $t < 0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t - \frac{|\vec{x} - \vec{x}'|}{c}\right] > 0$. Hence if $t < 0$, $\int d^3x' dt' G^+(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{in}}(\vec{x}, t)$.

.74

- Instead the $-$ case corresponds to the anticipated solution, with an exit condition Ψ_{out} (after the sources f are deactivated, at $t = 0$, so $f(\vec{x}, t) = 0$ if $t > 0$),

$$\Psi^-(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t) + \int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t'), \quad (\text{III.13})$$

where we see that if $t > 0$, there is no $\vec{x}$ for any given $\vec{x}'$ such that $\left[t + \frac{|\vec{x} - \vec{x}'|}{c}\right] < 0$. Hence if $t > 0$, $\int d^3x' dt' G^-(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') = 0$, and $\Psi(\vec{x}, t) = \Psi_{\text{out}}(\vec{x}, t)$.

- In general we use the retarded solution Ψ^+ .

.75

III.1.2 Retarded Potentials

- We normally use the retarded solution, with initial condition $\Psi_{\text{in}} = 0$, or, in compact notation,

$$\Psi(\vec{x}, t) = \int d^3x' \frac{[f(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.14})$$

where $[(\dots)]_{\text{re}}$ means to evaluate in $t' = t - |\vec{x} - \vec{x}'|/c$.

- Applying to the electrodynamic potentials,

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{[\rho(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.15})$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\vec{J}(\vec{x}', t')]_{\text{ret}}}{|\vec{x} - \vec{x}'|}. \quad (\text{III.16})$$

.76

III.1.3 Retarded electromagnetic field

- In order to calculate $\vec{E}$ and $\vec{B}$, we use $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$.
- Alternatively, we can use the Maxwell equations to reach:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0} \left(-\vec{\nabla}\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right), \quad (\text{III.17})$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J}. \quad (\text{III.18})$$

- Using the Green function, we get

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{R} \left[-\vec{\nabla}'\rho - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}}, \quad (\text{III.19})$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[\vec{\nabla}' \times \vec{J} \right]_{\text{ret}}. \quad (\text{III.20})$$

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- The expressions for the retarded fields Eqs. III.19 and III.21 can also be written in a form that connects directly with the static expressions (Jackson 6.5, **tarea**):

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} [\rho(\vec{x}', t')]_{\text{ret}} + \frac{\hat{R}}{cR} \left[\frac{\partial \rho(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \right\} \quad (\text{III.21})$$

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \left[\vec{J}(\vec{x}', t') \right]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\} \quad (\text{III.22})$$

.78

III.2 Multipolar Radiation

III.2.1 Wave zone

- This section follows Jackson Chap. 9.
- We now consider harmonic sources (the general case can be obtained by superposition of such sources):

$$\begin{aligned} \rho(\vec{x}, t) &= \rho(\vec{x}) e^{-i\omega t}, \\ \vec{J}(\vec{x}, t) &= \vec{J}(\vec{x}) e^{-i\omega t}. \end{aligned} \quad (\text{III.23})$$

- We saw that in the presence of sources, the field $\vec{A}(\vec{x}, t)$ generated in vacuum, and without spatial boundaries, is

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \int dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta \left(t' + \frac{|\vec{x} - \vec{x}'|}{c} - t \right).$$

- For harmonic sources,

$$\vec{A}(\vec{x}, t) = e^{-i\omega t} \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}, \quad (\text{III.24})$$

with $k = \omega/c$

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- We then obtain $\vec{H}$ and $\vec{E}$ with

$$\vec{H} = \frac{1}{\mu_o} \vec{\nabla} \times \vec{A}, \quad (\text{III.25})$$

and Faraday's law,

$$\vec{E} = \frac{i}{k} \sqrt{\frac{\mu_o}{\epsilon_o}} \vec{\nabla} \times \vec{H}. \quad (\text{III.26})$$

- We now consider sources confined inside a region whose maximum extension is d , and that contains the origin. If $d \ll \lambda$, there are 3 regions of interest:
 - The near zone, with $d < r \ll \lambda$, where $e^{ik|\vec{x}-\vec{x}'|} \sim 1$ and we recover the static potentials except for harmonic oscillation, $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$.
 - The intermediate zone with $d \ll r \sim \lambda$.
 - The far zone, with $d \ll r$.

.80

- In the far zone, with $d \ll r$, $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$, where $\hat{n} = \vec{x}/r \Rightarrow$

$$\lim_{kr \rightarrow \infty} \vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') e^{-ik\hat{n} \cdot \vec{x}'} d^3x'. \quad (\text{III.27})$$

- We see that $\vec{A}(\vec{x}, t) = \vec{A}(\vec{x})e^{-i\omega t}$ represents a spherical wave travelling outwards.
- In addition, (tarea) using Eqs III.25 and III.26 we also see that $\vec{E}$ and $\vec{H}$ also form spherical transverse waves (orthogonal to $\hat{n}$).
- The far zone thus corresponds to the *radiation zone*, also called *wave zone*.

.81

III.2.2 Dipolar radiation

- We now use $d \ll \lambda$ to simplify $\vec{A}$ in the wave zone. The integrand in Eq. III.27 can be expanded in powers of $-ik\hat{n} \cdot \vec{x}'$, using

$$e^{-ik\hat{n} \cdot \vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n.$$

- Therefore,

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}')^n d^3x'. \quad (\text{III.28})$$

.82

- For $n = 0$, which is the dominant term in the expansion in $k\hat{n} \cdot \vec{x}'$, we get:

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}') d^3x'. \quad (\text{III.29})$$

- Using the continuity equation, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, we have $i\omega\rho = \vec{\nabla} \cdot \vec{J}$.
- Therefore (tarea):

$$\int \vec{J}(\vec{x}') d^3x' = - \int \vec{x}' (\vec{\nabla}' \cdot \vec{J}) d^3x' = -i\omega \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (\text{III.30})$$

- Finally,

$$\vec{A}(\vec{x}) = \frac{-i\mu_o\omega}{4\pi} \underbrace{\vec{p}}_{\text{electric dipole}} \frac{e^{ikr}}{r}, \quad \text{with } \vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'. \quad (\text{III.31})$$

.83

- We now calculate the $\vec{E}$ and $\vec{H}$ fields:

$$\begin{aligned} \vec{H} &= \frac{ck^2}{4\pi} (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r}, \\ \vec{E} &= \sqrt{\frac{\mu_o}{\epsilon_o}} \vec{H} \times \hat{n}, \end{aligned} \quad (\text{III.32})$$

where we see that electric dipole radiation is linearly polarized.

- The power emitted in direction $\hat{n}$ can be written with $dP = r^2 d\Omega \hat{n} \cdot \vec{S} \Rightarrow$,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2} \Re \left[r^2 \hat{n} \cdot (\vec{E} \times \vec{H}^*) \right], \\ &= \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_o}{\epsilon_o}} k^4 |\vec{p}|^2 \sin^2(\theta). \end{aligned} \quad (\text{III.33})$$

- The total power is

$$P = \frac{c^2 k^4}{12\pi} \sqrt{\frac{\mu_o}{\epsilon_o}} |\vec{p}|^2. \quad (\text{III.34})$$

.84

III.2.3 Magnetic dipole and electric quadrupole radiation

- The next term in the expansion of $e^{-ik\hat{n} \cdot \vec{x}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} (\hat{n} \cdot \vec{x}')^n$ corresponds to $n = 1$.
- Eq. III.28 gives:

$$\vec{A}(\vec{x}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) \int \vec{J}(\vec{x}') (\hat{n} \cdot \vec{x}') d^3x'. \quad (\text{III.35})$$

- This term originates *magnetic dipole* and *electric quadrupole* contributions. To see this, we separate the integrand:

$$\vec{J}(\vec{x}')(\hat{n} \cdot \vec{x}') = \underbrace{\frac{1}{2} \left[(\hat{n} \cdot \vec{x}') \vec{J} + (\hat{n} \cdot \vec{J}) \vec{x}' \right]}_A + \underbrace{\frac{1}{2} (\vec{x}' \times \vec{J}) \times \hat{n}}_B. \quad (\text{III.36})$$

- We first consider the contribution of part B and identify the magnetization $\vec{\mathcal{M}}$,

$$\vec{\mathcal{M}} = \frac{1}{2} (\vec{x} \times \vec{J}). \quad (\text{III.37})$$

- Then, for the B part,

$$\vec{A}(\vec{x}) = \frac{ik\mu_o}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right), \quad \text{with} \quad (\text{III.38})$$

$$\vec{m} = \int \vec{\mathcal{M}} d^3x. \quad (\text{III.39})$$

- In the radiation zone, $kr \gg 1$, we obtain:

$$\vec{A}(\vec{x}) = \frac{ik\mu_o}{4\pi} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (\text{III.40})$$

$$\vec{E}(\vec{x}) = -\frac{k^2}{4\pi} \sqrt{\frac{\mu_o}{\epsilon_o}} (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}, \quad (\text{III.41})$$

$$\vec{H}(\vec{x}) = -\sqrt{\frac{\epsilon_o}{\mu_o}} (\vec{E} \times \hat{n}). \quad (\text{III.42})$$

- This contribution is called magnetic dipole radiation.

- For the A part in the contribution from $n = 1$, after standard handling we get:

$$A = \frac{i\omega}{2} \int \vec{x}' (\hat{n} \cdot \vec{x}') \rho(\vec{x}') d^3x', \quad (\text{III.43})$$

which represents order 2 moments of $\rho(\vec{x}')$, i.e. an electric quadrupole contribution, which we will not develop.

III.2.4 Spectrum of continuous dipole radiation

- Consider a time-dependent electric dipole $\vec{p}(t)$, for simplicity with constant direction. We decompose $\vec{p} = \int e^{-i\omega t} \vec{p}(\omega) d\omega$.
- For a single harmonic component,

$$\vec{p}(t) = e^{-i\omega t} \vec{p}(\omega), \quad (\text{III.44})$$

the radiated $\vec{E}(\omega)$ field is (from Eq. III.32),

$$\vec{E}(\vec{x}, \omega) = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{ck^2}{4\pi} \frac{e^{ikr}}{r} \hat{r} \times \vec{p}(\omega) \times \hat{r}, \quad (\text{III.45})$$

$$\vec{E}(\vec{x}, \omega) = \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \sin(\Theta) |p(\omega)| \hat{\Theta}, \quad (\text{III.46})$$

where Θ is co-latitude with $\hat{z} \parallel \vec{p}$.

.88

III.2.4.1 Continuous Spectra and Power Spectral Density

- The time-averaged flux radiated in direction $\hat{r}$ is $\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$ (Eq. II.36).
- However, each Fourier component is in fact a Fourier density, with contributions $d\vec{p} = e^{-i\omega t} \vec{p}(\omega) d\omega$. Directly summing these contributions would result in $d\omega^2$ in the radiated power.
- For steady-state sources, another problem in the direct superposition of Fourier components is that the standard Fourier transform diverges:

$$\vec{p}(\omega) = \int_{-\infty}^{\infty} \vec{p}(t) e^{i\omega t} dt \rightarrow \infty. \quad (\text{III.47})$$

.89

- We can still introduce the concept of a spectrum in an averaged sense. We define the truncated Fourier transform:

$$\vec{p}_T(\omega) = \int_{-T/2}^{T/2} \vec{p}(t) e^{i\omega t} dt, \quad (\text{III.48})$$

and the time average of its amplitude:

$$\langle |p|^2 \rangle(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} |\vec{p}_T(\omega)|^2. \quad (\text{III.49})$$

- The differential power for a continuous source becomes:

$$\frac{dP}{d\omega d\Omega} = \frac{\omega^4}{32\pi^2 \epsilon_0 c^3} \langle |p|^2 \rangle(\omega) \sin^2 \Theta. \quad (\text{III.50})$$

.90

III.3 Radiation from a single charge

III.3.1 Liénard-Wiechert potentials

- Consider a particle with charge q and trajectory $\vec{r}(t)$, with velocity $\vec{u}(t)$.
- We can apply the retarded solution with source terms $\rho(\vec{x}, t) = q\delta(\vec{x} - \vec{r}(t))$, and $\vec{j}(\vec{x}, t) = \rho(\vec{x}, t)\vec{u}(t)$, and thus obtain the resulting potentials (**tarea**):

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (\text{III.51})$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \left[\frac{q\vec{u}}{\left(1 - \hat{n}(t') \cdot \vec{\beta}(t')\right) R(t')} \right]_{\text{ret}}, \quad (\text{III.52})$$

where $\vec{R}(t') = \vec{x} - \vec{r}(t')$, $R = |\vec{R}|$, $\hat{n}(t') = \frac{\vec{R}}{R}$, and $\vec{\beta}(t') = \frac{\vec{u}(t')}{c}$. These are the Liénard-Wiechert (L.-W.) potentials.

.91

- We can also calculate the corresponding electromagnetic field (**tarea**):

$$\begin{aligned} \vec{E}(\vec{x}, t) = & \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{(1 - \beta^2)(\hat{n} - \vec{\beta})}{R^2(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{vel}}} \\ & + \underbrace{\left[\frac{q}{4\pi\epsilon_0} \frac{\hat{n} \times ((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}})}{cR(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{\text{ret}}}_{E_{\text{rad}}} \end{aligned} \quad (\text{III.53})$$

with

$$\vec{B}(\vec{x}, t) = \frac{1}{c} \hat{n} \times \vec{E}(\vec{x}, t). \quad (\text{III.54})$$

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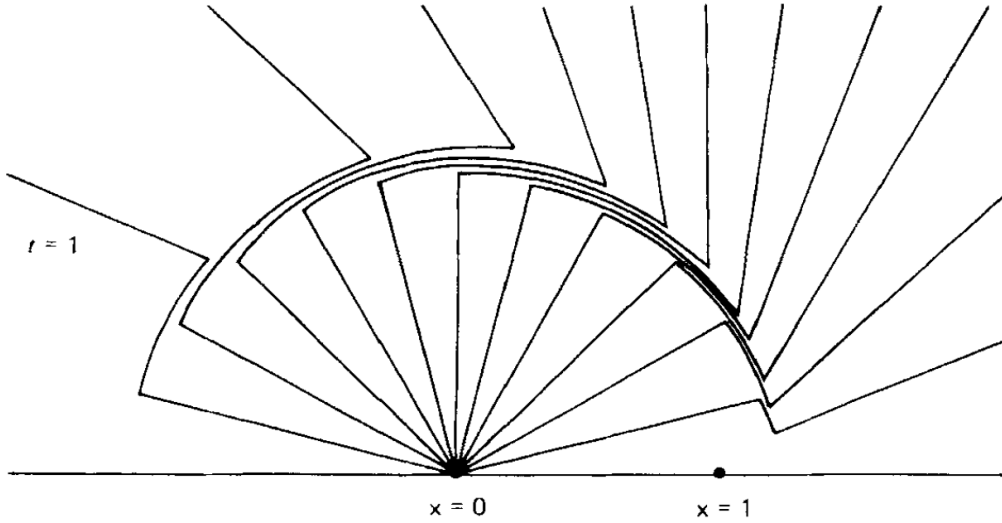
- We see that far away from the particle, the term labelled E_{rad} will eventually dominate. In fact, for a Fourier component, or for harmonic motion with $\vec{r}(t) \propto \exp(i\omega t)$, we find that (**tarea**)

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} = \beta \frac{R}{\lambda}. \quad (\text{III.55})$$

The radiation term dominates if $R \gg \lambda/\beta$, sometimes also called the *far zone*.

.93

- Illustration of the radiation field (from Rybicky & Lightman). A charge stops at $x = 0$ and $t = 0$. The figure shows the field lines at $t = 1$, where the charge would have been at $x = 1$ had it not stopped.



.94

III.3.2 Larmor Formula

- In the far zone, with $R \rightarrow \infty$, and in the Galilean limit ($\beta \ll 1$), the power radiated per unit solid angle is (tarea):

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} q^2 a^2 \sin^2(\theta), \quad (\text{III.56})$$

in which $a = |\dot{\vec{u}}|$ and θ is the angle between $\hat{n}$ and $\vec{a}$.

- By applying Eq. III.56 to the case of a harmonically oscillating charge with dipole $\vec{p} = q\vec{r}_0 \exp(i\omega t)$, we can recover Eq. III.33 (tarea).
- We see that the *wave zone* as defined in Sec. III.2.1 matches the Galilean limit (see the discussion on the dipole approximation in Sec. 3.3 of Rybicky & Lightman).

.95

- Integrating Eq. III.56 over all directions, we have the total power in SI:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}, \quad (\text{III.57})$$

which is the Larmor formula.

- In CGS,

$$P = \frac{2q^2 a^2}{3c^3}. \quad (\text{III.58})$$

.96

III.3.3 Dipole of a charge assembly

- Consider an assembly of charges confined within a region of size d (here we follow Rybicki & Lightman 3.3).
- If τ is the minimum timescale over which particle velocities $\langle v \rangle$ change appreciably, we can ignore differences in retarded times if $c\tau \gg d$.
- If $l < d$ is the scale associated to a trajectory, then $\frac{\langle v \rangle}{c} \ll \frac{l}{d}$, or $\langle v \rangle \ll c$.
- Therefore, if $c\tau \gg d$ (equivalent to $\lambda \gg d$ for harmonically oscillating charges), we take the Galilean limit $\beta \rightarrow 0$ and the radiation field of a charge distribution is

$$\vec{E}_{\text{rad}} = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\hat{n} \times (\hat{n} \times \dot{\vec{\beta}}_i)}{c|\vec{x} - \vec{x}_i|}. \quad (\text{III.59})$$

- To zeroth order,

$$\vec{E}_{\text{rad}}(x) = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{p}})}{4\pi\epsilon_0 c^2 r}. \quad (\text{III.60})$$

- The differential power is

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{p}}|^2}{(16\pi^2\epsilon_0 c^3)} \sin^2(\Theta), \quad (\text{III.61})$$

with $\hat{z} \parallel \vec{p}$ and where Θ is colatitude. In CGS,

$$\frac{dP}{d\Omega} = \frac{|\ddot{\vec{p}}|^2}{4\pi c^3} \sin^2(\Theta), \quad (\text{III.62})$$

- The total radiated power is

$$P = \frac{1}{6\pi\epsilon_0 c^3} |\ddot{\vec{p}}|^2. \quad (\text{III.63})$$

and in CGS,

$$P = \frac{2}{3} \frac{|\ddot{\vec{p}}|^2}{c^3} \quad (\text{III.64})$$

- In general, for a time-dependent dipole $\vec{p}(t)$ oscillating in a constant direction, Eq. III.60 gives

$$\vec{E}_{\text{rad}}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2 r} |\ddot{\vec{p}}(t)| \sin(\Theta), \quad (\text{III.65})$$

in SI, or, in CGS,

$$\vec{E}_{\text{rad}}(\vec{x}, t) = \frac{1}{c^2 r} |\ddot{\vec{p}}(t)| \sin(\Theta), \quad (\text{III.66})$$

with $p_{\text{SI}} = \sqrt{4\pi\epsilon_0} p_{\text{CGS}}$ and $E_{\text{SI}} = E_{\text{CGS}}/\sqrt{4\pi\epsilon_0}$.

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III.3.4 Radiation spectrum of a pulse

- Consider a “pulse” $\vec{E}(t)$, with finite time support. The spectrum,

$$\vec{E}(\omega) = \frac{1}{2\pi} \int \vec{E}(t) e^{i\omega t} dt, \quad (\text{III.67})$$

is Hermitian, $\vec{E}(-\omega) = \vec{E}^*(\omega)$, for $\vec{E}(t)$ to be real.

- The power radiated per unit area is given by the Poynting vector,

$$\frac{dW}{dt dA} = |\vec{S}(t)| = \frac{c}{4\pi} |\vec{E}(t)|^2 \text{ in CGS.} \quad (\text{III.68})$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} |\vec{E}(t)|^2 \text{ in SI.} \quad (\text{III.69})$$

- In CGS, the energy radiated by the pulse is

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} |\vec{E}(t)|^2 dt. \quad (\text{III.70})$$

.100

- From Parseval's theorem,

$$\int |\vec{E}(t)|^2 dt = 2\pi \int |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.71})$$

and with the Hermitian property,

$$\int |\vec{E}(t)|^2 dt = 4\pi \int_0^{\infty} |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.72})$$

- Therefore, in CGS

$$\frac{dW}{dA} = c \int_0^{\infty} |\vec{E}(\omega)|^2 d\omega, \quad (\text{III.73})$$

and the spectrum of the energy radiated by the pulse is

$$\frac{dW}{dA d\omega} = c |\vec{E}(\omega)|^2. \quad (\text{III.74})$$

- For SI,

$$\frac{dW}{dA d\omega} = 4\pi \sqrt{\frac{\epsilon_0}{\mu_0}} |\vec{E}(\omega)|^2. \quad (\text{III.75})$$

.101

- We can revisit the spectrum of dipole radiation with

$$\vec{p} = \int e^{-i\omega t} \vec{p}(\omega) d\omega. \quad (\text{III.76})$$

- Since $\ddot{\vec{p}}(t) = - \int \omega^2 \vec{p}(\omega) e^{-i\omega t} d\omega$, in CGS

$$\vec{E}(\omega) = -\frac{1}{c^2 r} \omega^2 \vec{p}(\omega) \sin(\Theta). \quad (\text{III.77})$$

- From Eq. III.74, the spectrum for the radiated energy is therefore, in CGS,

$$\frac{dW}{d\omega d\Omega} = \frac{1}{c^3} \omega^4 |\vec{p}(\omega)|^2 \sin^2 \Theta. \quad (\text{III.78})$$

- The total radiated energy, in CGS, is

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3} \omega^4 |\vec{p}(\omega)|^2. \quad (\text{III.79})$$

.102

- In SI, the spectrum for the angular-differential radiated energy is

$$\frac{dW}{d\omega d\Omega} = \frac{1}{4\pi\epsilon_0 c^3} \omega^4 |\vec{p}(\omega)|^2 \sin^2 \Theta, \quad (\text{III.80})$$

which can be compared to the dipole approximation for continuous sources, which, from Eq. III.50, is

$$\frac{dP}{d\omega d\Omega} = \frac{1}{8\pi} \frac{1}{4\pi\epsilon_0 c^3} \omega^4 \langle |p|^2 \rangle(\omega) \sin^2(\Theta). \quad (\text{III.81})$$

in SI. The extra factor $1/(8\pi)$ in the power of continuous sources comes from the time-average (1/2), and from the application of Parseval's theorem for the total pulse energy ($1/(4\pi)$). Also, Eq. III.80 refers to the radiated energy, with $\vec{p}(\omega)$ as a Fourier density in C m s (see Eq. III.76), whereas Eq. III.81 refers to the radiated power of a continuous source, with time-averaged squared amplitude $\langle |p|^2 \rangle(\omega)$ (Eq. III.49), in units of C² m² s.

.103

III.3.5 Radiation reaction

- This section is adapted to SI from Rybicki & Lightman Sec. 3.5 (see also Jackson Chap. 16 or Griffiths Sec. 11.2.2).
- Consider a periodic system, such that its mechanical state is identical between times t_1 and t_2 .
- Still in the Galilean limit, the total energy radiated by the charge between t_1 and t_2 is (in SI),

$$W = \frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2 dt. \quad (\text{III.82})$$

.104

- Energy conservation requires the existence of a ‘radiation reaction’ force F_{rad} , such that W is extracted from the particle’s kinetic energy,

$$\int_{t_1}^{t_2} dt \vec{F}_{\text{rad}} \cdot \vec{u} = -W. \quad (\text{III.83})$$

- One expression for the radiation reaction is the Abraham-Lorentz formula,

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}. \quad (\text{III.84})$$

We can confirm that Eq. III.84 fulfills Eq. III.83 (tarea).

- In CGS,

$$\vec{F}_{\text{rad}} = \frac{2q^2}{3c^3} \dot{\vec{a}} = m \tau \dot{\vec{a}}, \quad (\text{III.85})$$

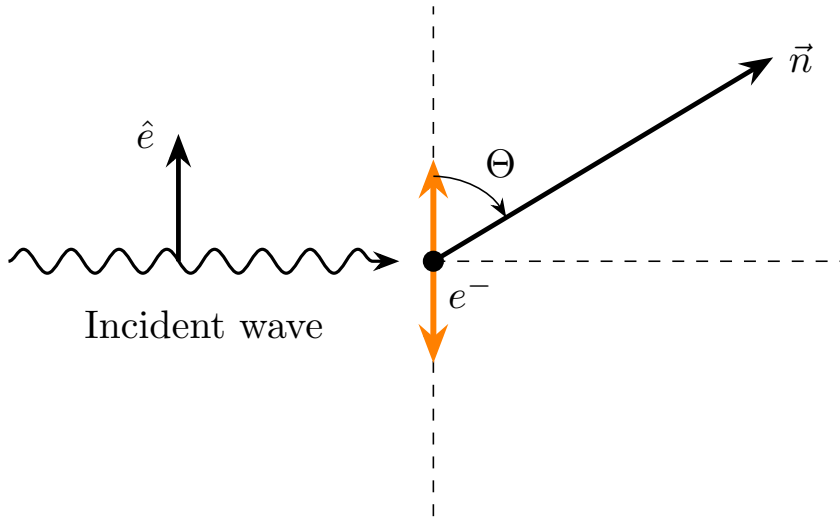
where $\tau = \frac{2}{3} r_0 / c$.

.105

III.4 Non-relativistic applications

III.4.1 Thomson scattering

This section follows from Rybicki & Lightman Sec. 3.4.



- A linearly polarized wave exerts the force $\vec{F} = q\hat{e}E_0 \sin(\omega t)$ on a charge q .

.106

- The equation of motion

$$m\ddot{\vec{r}} = \hat{e}E_o \sin(\omega t), \quad (\text{III.86})$$

yields the dipole $\vec{d} = \hat{e}\vec{r}$ from integration of the acceleration

$$\vec{d} = q^2 \hat{e} \frac{E_o}{m} \sin(\omega t), \quad (\text{III.87})$$

and

$$\vec{d} = - \left(\frac{q^2 E_o}{m\omega_o^2} \right) \hat{e} \sin(\omega t). \quad (\text{III.88})$$

.107

- The amplitude of the oscilating dipole is $d = \frac{q^2 E_o}{m\omega^2}$.
- Dipolar radiation (Eq. III.33) yields, in SI (or MKS),

$$\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_o}{\epsilon_o}} k^4 |\vec{d}|^2 \sin^2(\Theta). \quad (\text{III.89})$$

- In CGS, the same power is (Eq. 3.23 in Rybicki & Lightman, with an additional 1/2 for time-averaging):

$$\frac{dP}{d\Omega} = \frac{1}{c} \frac{k^4 d^2}{8\pi} \sin^2(\Theta). \quad (\text{III.90})$$

The conversion can be obtained by substitution of $\vec{d}_{\text{CGS}} = \vec{d}_{\text{SI}}/\sqrt{4\pi\epsilon_0}$ and $c = 1/\sqrt{\epsilon_0\mu_0}$.

.108

- The incident power is

$$\langle \vec{S} \rangle = \frac{c}{8\pi} E_o^2 \hat{n}, \quad (\text{III.91})$$

in CGS, and

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathcal{E}|^2 \hat{n}, \quad (\text{III.92})$$

in SI (Eq. II.36).

- We introduce the differential cross section

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega}. \quad (\text{III.93})$$

.109

- The differential cross section for Thomson scattering is

$$\frac{d\sigma}{d\Omega} = r_o^2 \sin^2(\Theta), \quad (\text{III.94})$$

where

$$r_o = \frac{e^2}{m c^2} \quad (\text{III.95})$$

is the classical radius of the electron ($r_o \sim 3 \times 10^{-13}$ cm) in CGS, and

$$r_o = \frac{e^2}{4\pi\epsilon_0 m c^2} \quad (\text{III.96})$$

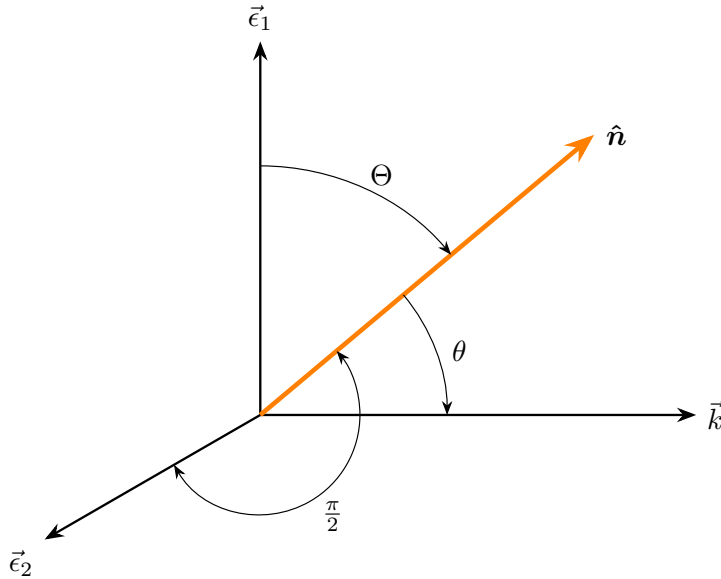
in SI.

- The total cross section is

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} r_o^2. \quad (\text{III.97})$$

.110

- We can extend the previous derivation to the case of unpolarized radiation.
- An unpolarized beam is the sum of two independent linearly polarized components, along $\hat{e}_1$ and $\hat{e}_2$.
- We place $\hat{e}_1$ in the plane of scattering (see Fig.).



.111

- For unpolarized incident radiation, the Poynting flux of each of two components of the incident field, E_1 and E_2 , are equal.
- The scattered field is the sum of the scattered fields of each component,

$$\vec{E}_{\text{scat}}(t) = \vec{E}_{1,\text{scat}}(t) + \vec{E}_{2,\text{scat}}(t), \quad (\text{III.98})$$

and

$$\begin{aligned} \left\langle \left| \vec{E}_{\text{scat}}(t) \right|^2 \right\rangle &= \left\langle \left| \vec{E}_{1,\text{scat}}(t) \right|^2 \right\rangle + \left\langle \left| \vec{E}_{2,\text{scat}}(t) \right|^2 \right\rangle + \\ &\quad \left\langle \vec{E}_{1,\text{scat}}(t) \cdot \vec{E}_{2,\text{scat}}^*(t) \right\rangle + \left\langle \vec{E}_{1,\text{scat}}^*(t) \cdot \vec{E}_{2,\text{scat}}(t) \right\rangle, \end{aligned} \quad (\text{III.99})$$

where the cross-terms are null.

.112

- Therefore,

$$\langle dP \rangle = \langle dP_1 \rangle + \langle dP_2 \rangle, \quad (\text{III.100})$$

and, in terms of the total incident Poynting field, the total cross section is the average of the cross-sections for each linear polarization components,

$$\left. \frac{d\sigma}{d\Omega}(\Theta) \right|_{\text{unpol}} = \frac{1}{2} \left[\left. \frac{d\sigma}{d\Omega}(\Theta) \right|_{\text{pol}} + \left. \frac{d\sigma}{d\Omega} \left(\frac{\pi}{2} \right) \right|_{\text{pol}} \right], \quad (\text{III.101})$$

$$= \frac{1}{2} r_0^2 (1 + \sin^2 \Theta), \quad (\text{III.102})$$

$$= \frac{1}{2} r_0^2 (1 + \cos^2 \theta). \quad (\text{III.103})$$

$$(\text{III.104})$$

.113

- The total scattered intensity (i.e. total power) is proportional to $1 + \cos^2 \theta$. From this intensity the weaker component (in the plane of scattering) has intensity proportional to $\cos^2 \theta$. The unpolarized scattered intensity must have equal components, and is proportional to $2 \cos^2 \theta$. The polarized scattered intensity is thus proportional to $1 + \cos^2 \theta - 2 \cos^2 \theta$.
- The polarization fraction is

$$\Pi = \frac{1 - \cos^2(\theta)}{1 + \cos^2(\theta)}. \quad (\text{III.105})$$

.114

III.4.2 Harmonically bound particles

- This section follows from Rybicki & Lightman Sec. 3.6, and uses CGS units.
- The equation of motion for a radiating harmonic oscillator is

$$-\tau \ddot{x} + \ddot{x} + \omega_0 x = 0, \quad (\text{III.106})$$

where τ in the radiation reaction force is given by Eq. III.85.

- We assume that the damping due to radiation will be small, and use $x(t) \propto \cos(\omega_0 t + \phi_0)$, so $\ddot{x} \approx -\omega_0^2 \dot{x}$.
- The equation of motion becomes

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0. \quad (\text{III.107})$$

.115

- Searching for solutions to Eq. III.107 in the form $e^{\alpha t}$, we require

$$\alpha \approx \pm i\omega_0 - \frac{1}{2}\omega_0^2 \tau, \quad (\text{III.108})$$

if $\omega_0 \tau \ll 1$.

- With the initial condition $x(0) = x_0$ and $\dot{x}(0) = 0$,

$$x(t) = x_0 e^{-\Gamma \frac{t}{2}} \cos(\omega_0 t), \quad (\text{III.109})$$

$$= \frac{1}{2} x_0 \left(e^{-\frac{1}{2}\Gamma t + i\omega_0 t} + e^{-\frac{1}{2}\Gamma t - i\omega_0 t} \right), \quad (\text{III.110})$$

where $\Gamma = \omega_0^2 \tau$.

.116

- The Fourier transform of $x(t)$ is

$$x(\omega) = \frac{1}{2\pi} \int x(t) e^{i\omega t} dt = \frac{x_0}{4\pi} \left[\frac{1}{\Gamma/2 - i(\omega + \omega_0)} + \frac{1}{\Gamma/2 - i(\omega - \omega_0)} \right]. \quad (\text{III.111})$$

- Restricting to positive frequencies, and neglecting the tail from the negative singularity,

$$x(\omega) \approx \frac{x_0}{4\pi} \frac{1}{\Gamma/2 - i(\omega - \omega_0)}, \quad (\text{III.112})$$

whose squared modulus is

$$|x(\omega)|^2 \approx \left(\frac{x_0}{4\pi} \right)^2 \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0)^2}. \quad (\text{III.113})$$

.117

- From Eq. III.79, the radiated energy spectrum is then

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} \frac{e^2 x_0^2}{(4\pi)^2} \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0)^2}. \quad (\text{III.114})$$

- With $k \equiv m\omega_0^2$ (spring constant) and $\Gamma = \omega_0^2\tau$,

$$\frac{dW}{d\omega} = \left(\frac{1}{2} k x_0^2 \right) \frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2} = \epsilon_k \phi(\omega), \quad (\text{III.115})$$

where ϵ_k is the potential energy of the spring, and $\phi(\omega)$ is the profile of the spectrum, with $\int \phi(\omega) d\omega = 1$.

- The profile $\phi(\omega)$ is the Lorentz profile.

.118

- Now consider a driven harmonically bound particle, such as a charge under the influence of a linearly polarized plane wave,

$$m\ddot{x} = -m\omega_0^2 x + m\tau\ddot{x} + eE_0 \cos(\omega t). \quad (\text{III.116})$$

- The steady solution is $x = x_0 e^{i\omega t} \equiv |x_0| e^{i(\omega t + \delta)}$, with

$$x_0 = -\frac{e E_0}{m} / (\omega^2 - \omega_0^2 - i\omega_0^3\tau), \quad (\text{III.117})$$

$$\delta = \arctan \left(\frac{\omega^2\tau}{\omega^2 - \omega_0^2} \right). \quad (\text{III.118})$$

- In steady state, the charge acts as an oscillating dipole, radiating a total power given by the Larmor formula (Eq. III.58 or Eq. III.64):

$$P = \frac{e^2 |x_0|^2 \omega^4}{3c^3} = \frac{e^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3\tau)^2}. \quad (\text{III.119})$$

.119

- Comparing the radiate power from Eq. III.119 with the incident time-average flux, $\langle S \rangle = \frac{c}{8\pi} E_0^2$, we introduce the scattering cross section

$$\sigma(\omega) = \frac{P(\omega)}{\langle S \rangle} = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + (\omega_0^3\tau)^2}. \quad (\text{III.120})$$

- In the limit $\omega \gg \omega_0$, we recover Thomson scattering.
- In the limit $\omega \ll \omega_0$, we enter the dipolar approximation regime and

$$\sigma(\omega) = \sigma_T \left(\frac{\omega}{\omega_0} \right)^4. \quad (\text{III.121})$$

This regime is known as Rayleigh Scattering.

.120

- In the regime $\omega \sim \omega_0$ (tarea),

$$\sigma(\omega) \approx \frac{2\pi^2 e^2}{m c} \frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2}, \quad (\text{III.122})$$

where we recognize the Lorentz profile.

- If we integrate this approximation over all frequencies $\nu = \omega/(2\pi)$,

$$\int_0^\infty d\nu \sigma(\nu) = \frac{\pi e^2}{m c}. \quad (\text{III.123})$$

- In the quantum mechanical calculation of radiative cross sections between two state n_1 and n_2 , the resulting cross sections are sometimes referred to the classical treatment of Eq. III.123 through the oscillator strength $f_{n_1 n_2}$:

$$\int_0^\infty d\nu \sigma(\nu) = f_{n_1 n_2} \frac{\pi e^2}{m c}. \quad (\text{III.124})$$

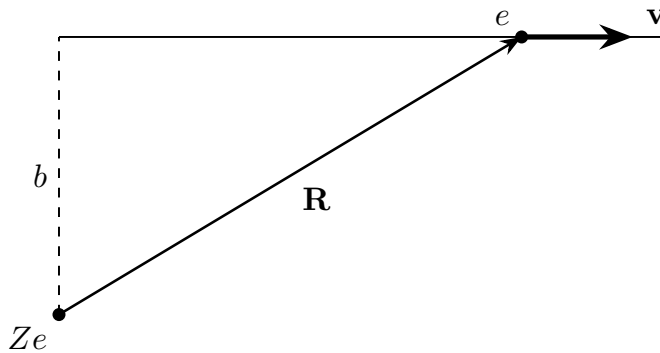
.121

III.4.3 Thermal free-free

- This section follows from Rybicki & Lightman Secs. 5.1-5.3.
- In an ionized plasma, radiation stems from electron-proton encounters, with dipole $\vec{d} = -e\vec{R}$, with acceleration $\ddot{\vec{d}} = -e\ddot{\vec{v}}$.
- In the Fourier plane,

$$-\omega^2 \vec{d}(\omega) = -\frac{e}{2\pi} \int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t} dt. \quad (\text{III.125})$$

- We introduce the impact parameter b .



.122

- A timescale for the collision is the collision time $\tau \equiv \frac{b}{v}$.
- If $\omega\tau \gg 1$, the integrand in Eq. III.125 oscillates rapidly and averages to 0, while if $\omega\tau \ll 1$ this integrand is essentially unity:

$$\vec{d}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2} \Delta\vec{v}, & \text{if } \omega\tau \ll 1, \\ 0, & \text{if } \omega\tau \gg 1. \end{cases} \quad (\text{III.126})$$

- We can calculate the spectrum of the emitted radiation using Eq. III.79,

$$\frac{dW}{d\omega} = \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta\vec{v}|^2, & \omega\tau \ll 1, \\ 0, & \omega\tau \gg 1. \end{cases} \quad (\text{III.127})$$

.123

- We estimate $\Delta\vec{v}$, the velocity change in a collision, by assuming that the electron follows a straight line. The longitudinal acceleration cancels out on both sides of the collision. However, the perpendicular component pulls the electron all along its trajectory. We integrate this normal acceleration along a straight path (and neglecting the concomitant deflection of the path):

$$\Delta v = \frac{Ze^2}{m} \int_{-\infty}^{\infty} \frac{b dt}{(b^2 + v^2 t^2)^{3/2}} = \frac{2Ze^2}{mbv}. \quad (\text{III.128})$$

- The radiated energy is then

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2 e^6}{3\pi c^3 m^2 v^2 b^2}, & b \ll v/\omega \\ 0, & b \gg v/\omega. \end{cases} \quad (\text{III.129})$$

.124

- For a fixed electron speed v , we can sum over b to get the total emissivity:

$$\frac{dW}{d\omega dV dt} = n_e n_i 2\pi v \int_{b_{\min}}^{\infty} \frac{dW(b)}{d\omega} b db, \quad (\text{III.130})$$

starting from a minimum impact parameter $b_{\min}$.

- Since the integrand in Eq. III.130 is zero above a maximum impact parameter $b_{\max} = \frac{v}{\omega}$,

$$\begin{aligned} \frac{dW}{d\omega dV dt} &= \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} \\ &= \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \ln \left(\frac{b_{\max}}{b_{\min}} \right) \end{aligned} \quad (\text{III.131})$$

.125

- The value of $b_{\min}$ can be estimated as the smallest value for the straight-line trajectory approximation, i.e. $\Delta v \sim v$ and $b_{\min} = 4Ze^2/(\pi mv^2)$.
- The floor value of $b_{\min}$ is set by the uncertainty principle, with $\Delta x \Delta p \gtrsim \hbar$, $\Delta x \sim b$ and $\Delta p \sim mv$, or $b_{\min} = \hbar/(mv)$.
- An adequate treatment of the total radiated power requires field quantization, and is beyond the scope of this course. The connection with the classical treatment presented here is implemented through a Gaunt factor $g_{ff}(v, \omega)$, typically of order unity, and

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m^2 v} n_e n_i Z^2 g_{ff}(v, \omega), \quad (\text{III.132})$$

with

$$g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln \left(\frac{b_{\max}}{b_{\min}} \right). \quad (\text{III.133})$$

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- For a thermal distribution of speeds, we average Eq. III.132 over the Maxwellian distribution, with probability density

$$dP \propto v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv. \quad (\text{III.134})$$

- For radiation at frequency ν , we require $h\nu < \frac{1}{2}mv^2$ for photon creation,

$$\frac{dW(T, \omega)}{dV dt d\omega} = \frac{\int_{v_{\min}}^{\infty} \frac{dW(v, \omega)}{d\omega dV dt} v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv}{\int_0^{\infty} v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv}, \quad (\text{III.135})$$

and we obtain the emission coefficient (tarea):

$$j_{\nu}^{ff}(\nu) = \frac{1}{4\pi} \frac{dW}{dV dt d\nu} = \frac{2^5 \pi e^6}{3mc^3} \left(\frac{2\pi}{3km} \right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}. \quad (\text{III.136})$$

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- The associated absorption coefficient is give by Kirchof's law:

$$j_{\nu}^{ff} = \alpha_{\nu}^{ff} B_{\nu}(T). \quad (\text{III.137})$$

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